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The operational matrix of fractional derivative of the fractional-order Chebyshev functions and its applications

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Abstract In this paper, we introduce a family of fractional-order Chebyshev functions based on the classical Chebyshev polynomials. We calculate and derive the operational matrix of derivative of fractional order γ in the Caputo sense using the fractional-order Chebyshev functions. This matrix yields to low computational cost of numerical solution of fractional order differential equations to the solution of a system of algebraic equations. Several numerical examples are given to illustrate the accuracy of our method. The results obtained, are in full agreement with the analytical solutions and numerical results presented by some previous works.

Keywords. Chebyshev polynomials; orthogonal system; fractional differential equation; fractional-order Chebyshev functions; operational matrix.

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1. Introduction

Fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order, have been playing an increasingly important role in scientific and engineering computations. A history of the development of fractional differential operators and some applications can be found in [7, 9, 12, 23, 26, 35, 36]. In some cases, fractional differentials and integrals provide more accurate models of systems under consideration. The analytic results on existence and uniqueness of solutions to fractional differential equations (FDEs) have been investigated by many authors [9, 23]. Also analytic solution of some kinds of FDEs was investigated in [20]. Most FDEs do not have closed form solutions, so approximation and numerical techniques such as differential transform method [32], finite difference methods [1, 41], variational iteration method [11, 29], Adomian decomposition method [30], homotopy analysis method [10, 33], collocation method [18, 38, 40] and other methods [2, 4, 5, 13, 19, 24, 25, 27, 28, 43], must be used.

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Nowadays, a lot of attention has been devoted to construct operational matrix of fractional derivative for some types of classical orthogonal polynomials. For example, Darani et. al. in [8] introduced the fractional type of Chebysheve polynomials of the second kind and used it to numerical solution of some linear fractional differential equations. Saadatmandi and Dehghan [39] introduced shifted Legendre operational matrix for fractional derivatives and applied it for numerical solution of FDEs and fractional linear systems of equations [3]. Also, the Bernstein operational matrix of fractional derivative has been derived in [37]. Furthermore, Doha et al. [14] introduced shifted Chebyshev operational matrix for fractional derivatives and the authors in [15] derived the shifted Jacobi operational matrix of fractional derivatives. Application of classical orthogonal polynomials for the FDEs implies some difficulties in connection with the collocation method [18]. Recently, Kayedi-Bardeh et al. [21] introduced the fractional orthogonal Jacobi functions then they obtained the explicit form of the fractional derivative operational matrix for these functions. Also very recently, in [22] a general formulation for the fractional-order Legendre functions (FLFs) is constructed to obtain the solution of the FDEs. The methods based on operational matrices are the powerful tools in computational sciences[3, 4, 5, 14, 17, 24, 39]. In this paper, we introduce a new operational method to solve FDEs. The algorithms in the present work are somewhat related to the ideas used by Kayedi-Bardeh et al. [21] and Kazem et al. [22]. First, we construct the fractional order Chebyshev functions (FCFs) and then derive the operational matrix of fractional order FCFs and apply it to solve FDEs. The method reduces the FDEs to a system of algebraic equations.

The structure of this paper is arranged in the following way: In Section 2, we introduce some necessary definitions and mathematical preliminaries of fractional calculus. In Section 3, the FCFs and their properties are obtained. We make a new operational matrix for fractional derivative by FCFs in Section 4. Applications of the operational matrix are given in Section 5 and numerical simulations are reported in Section 6.

2. Preliminaries and notation

2.1. **A short overview on Chebyshev polynomials.** The Chebyshev polynomials of all kinds are widely use in approximation of functions [4, 14, 16, 31]. The well known Chebyshev polynomials of the first kind of degree n are defined on the interval $[-1, 1]$ as

$$T_n(t) = \cos(n \arccos(t)), \tag{2.1}$$

and the so-called shifted Chebyshev polynomials by using the simple change of variable are defined as

$$T^*_n(t) = T_n(2t-1). \tag{2.2}$$

Then $T^*_n(t)$ can be obtained with the aid of the following recurrence formula:

$$T^*_{n+1}(t) = (4t-2)T^*_n(t) - T^*_{n-1}(t), \qquad n = 1, 2, ..., \tag{2.3}$$

with initial conditions $T_0^*(t) = 1$ and $T_1^*(t) = 2t - 1$. The analytic form of the shifted Chebyshev polynomials $T_n^*(t)$ of degree $n > 0$ is given by

$$T_n^*(t) = n \sum_{k=0}^{n} (-1)^{(n-k)} \frac{(n+k-1)!2^{2k}}{(n-k)!(2k)!} t^k, \tag{2.4}$$

where $T_n^*(0) = (-1)^n$ and $T_n^*(1) = 1$. The orthogonality condition is

$$\int_0^1 T_m^*(t) T_n^*(t) w(t) = h_n \delta_{mn}, \tag{2.5}$$

where $w(t) = \frac{1}{\sqrt{t-t^2}}$, $h_n = \frac{b_n}{2}\pi$, $b_0 = 2, b_n = 1, n \geq 1$ and δ_{mn} is the Kronecker function.

Also, for the shifted Chebyshev polynomials of the first kind we have [31]

$$\mathbf{A}T = \begin{bmatrix} \frac{1}{2} \\ 2t \\ 8t^2 \\ \vdots \\ \frac{1}{2}(4t)^N \end{bmatrix}, \tag{2.6}$$

where

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 3 & 4 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{2N}{N} & \binom{2N}{N-1} & \binom{2N}{N-2} & \cdots & 1 \end{bmatrix}, \quad T = \begin{bmatrix} T_0^*(t) \\ T_1^*(t) \\ \vdots \\ T_N^*(t) \end{bmatrix}.$$

Using the above equations, we can write

$$T = \mathbf{F}X, \tag{2.7}$$

where $X = [1, t, t^2, \cdots, t^N]^T$ and $\mathbf{F} = \mathbf{A}^{-1}\mathbf{E}$ and $\mathbf{E} = (e_{ij})$ is the diagonal matrix with entire $e_{i,i} = \frac{1}{2}4^i, (i = 0, 1, \cdots, N)$. Thus, we can calculate each entire of T as

$$T_i^*(t) = \sum_{j=0}^{i} f_{ij} t^j, \quad i = 0, 1, 2, \cdots, N.$$

Remark 2.1. From 2.4 we get

$$A_{i,j}^{-1} = \begin{cases} (-1)^i & j = 0, \\ i(-1)^{i-j} \frac{(i+j-1)!2^{2j}}{(i-j)!(2j)!} & \text{otherwise} \end{cases}$$

2.2. **The fractional derivative in the Caputo sense.** Let us start with recalling the essentials of the fractional calculus. There are various definitions of fractional integration and differentiation of order $\gamma > 0$, and not necessarily equivalent to each other, (see, e.g. [23, 35]). The Caputo fractional derivative, which is used in this paper, allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physical interpretations.

Definition 2.2. *Caputo's definition of the fractional-order derivative is defined as*

$$D^{\gamma}f(x)=\frac{1}{\Gamma(n-\gamma)}\int_0^x\frac{f^{(n)}(t)}{(x-t)^{\gamma+1-n}}dt,\qquad n-1<\gamma<n,\quad n\in\mathbb{N},\tag{2.8}$$

where $\gamma>0$ is the order of the derivative, $\Gamma(.)$ is the Gamma function and $n=[\gamma]+1$, with $[\gamma]$ denoting the integer part of γ. Recall that for $\gamma\in\mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order. Similar to integer-order differentiation, Caputo's fractional differentiation is a linear operator:

$$D^{\gamma}(\lambda f(x)+\mu g(x))=\lambda D^{\gamma}f(x)+\mu D^{\gamma}g(x),\tag{2.9}$$

where λ and μ are constants. Also, for the Caputo's derivative we have [12],

$$D^{\gamma}C=0,\quad (C \text{ is a constant}),\tag{2.10}$$

$$D^{\gamma}x^{\alpha}=\begin{cases}0, & \text{for } \alpha\in\mathbb{N}_0 \text{ and } \alpha<\lceil\gamma\rceil,\\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\gamma)}x^{\alpha-\gamma}, & \text{for } \alpha\in\mathbb{N}_0 \text{ and } \alpha\geq\lceil\gamma\rceil \text{ or } \alpha\notin\mathbb{N} \text{ and } \alpha>\lfloor\gamma\rfloor,\end{cases}\tag{2.11}$$

where $\lceil\gamma\rceil$ and $\lfloor\gamma\rfloor$ are the ceiling and floor functions respectively. Also $\mathbb{N}=\{1,2,...\}$ and $\mathbb{N}_0=\{0,1,2,...\}$. Furthermore we need the generalized Taylor's formula that involves Caputo fractional derivatives. This generalization is presented in [34]:

Theorem 2.3. *(Generalized Taylor formula) Let $D^{i\alpha}f(x)\in C(0,1]$ for $i=0,1,\cdots N$ and $0<\alpha\leq 1$. Then*

$$f(x)=\sum_{i=0}^{N-1}\frac{x^{i\alpha}}{\Gamma(i\alpha+1)}D^{i\alpha}f(0^+)+R_N^{\alpha}(x)$$

where

$$R_N^{\alpha}(x)=\frac{x^{N\alpha}}{\Gamma(N\alpha+1)}D^{N\alpha}f(\xi_x),\quad \xi_x\in(0,x],\ \forall x\in(0,1],$$

and $D^{i\alpha}=\overbrace{D^{\alpha}D^{\alpha}\cdots D^{\alpha}}^{i\ times}$.

3. Fractional-order Chebyshev functions

Following [21, 22], the fractional-order Chebyshev functions (FCFs) can be defined by introducing the change of variable $t=x^{\alpha}$ and $\alpha>0$ on the shifted Chebyshev polynomials of the first kind. We denote the fractional-order Chebyshev functions $T_i^*(x^{\alpha})$ as $\overline{T}_i^{\alpha}(x)$. From the recurrence relation of the shifted Chebyshev polynomials (2.3), we find that $\overline{T}_i^{\alpha}(x)$ can be obtained with the following recurrence formula:

$$\overline{T}_{i+1}^{\alpha}(x)=(4x^{\alpha}-2)\overline{T}_i^{\alpha}(x)-\overline{T}_{i-1}^{\alpha}(x),\qquad i=0,1,2,...,\tag{3.1}$$

where $\overline{T}_0^{\alpha}(x) = 1$ and $\overline{T}_1^{\alpha}(x) = 2x^{\alpha} - 1$. Also, using Eq. (2.4), the analytic form of $\overline{T}_i^{\alpha}(x)$ of degree $i\alpha$ given by

$$\overline{T}_i^{\alpha}(x) = i\sum_{k=0}^{i}(-1)^{(i-k)}\frac{(i+k-1)!2^{2k}}{(i-k)!(2k)!}x^{k\alpha}. \tag{3.2}$$

In the following we state some properties of the FCFs.

Lemma 3.1. *$\overline{T}_i^{\alpha}(x)$s are orthogonal over the interval $(0,1)$ with respect to the weight function $w_{\alpha}(x) = \frac{1}{x\sqrt{x^{-\alpha}-1}}$ and we have:*

$$\int_0^1 \overline{T}_i^{\alpha}(x)\overline{T}_j^{\alpha}(x)w_{\alpha}(x)dx = \frac{1}{\alpha}h_i\delta_{ij},$$

Proof. Using Eq. (2.5) and taking $t = x^{\alpha}$, we get

$$\begin{aligned}\int_0^1 T_i^*(x^{\alpha})T_j^*(x^{\alpha})w(x^{\alpha})\alpha x^{\alpha-1}dx &= \int_0^1 \overline{T}_i^{\alpha}(x)\overline{T}_j^{\alpha}(x)\frac{\alpha x^{\alpha-1}}{\sqrt{x^{\alpha}-x^{2\alpha}}}dx\\ &= \int_0^1 \overline{T}_i^{\alpha}(x)\overline{T}_j^{\alpha}(x)\frac{\alpha x^{\alpha-1}}{x^{\alpha}\sqrt{x^{-\alpha}-1}}dx\\ &= \alpha\int_0^1 \overline{T}_i^{\alpha}(x)\overline{T}_j^{\alpha}(x)\frac{1}{x\sqrt{x^{-\alpha}-1}}dx\\ &= h_i\delta_{ij},\end{aligned}$$

this leads to the desired result. □

It is important to note that, both of x and $\sqrt{x^{-\alpha}-1}$ are positive, so $w_{\alpha}(x) = \frac{1}{x\sqrt{x^{-\alpha}-1}}$ is well defined.

Proposition 3.2. *The fractional-order Chebysev function $\overline{T}_i^{\alpha}(t)$, has precisely i zeros in the form*

$$t_j = \left(\frac{1}{2} + \frac{1}{2}\cos\left(\frac{(2j-1)\pi}{2i}\right)\right)^{\frac{1}{\alpha}}, \ j = 1, 2, \cdots, i. \tag{3.3}$$

Proof. The shifted Chebyshev polynomial $T_i^*(x)$ has precisely i zeros

$$x_j = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{(2j-1)\pi}{2i}\right), \ j = 1, 2, \cdots, i. \tag{3.4}$$

Thus $T_i^*(x)$ can be written as

$$T_i^*(x) = (x - x_1)(x - x_2)\cdots(x - x_i).$$

Taking the change of variable $x = t^{\alpha}$ we obtain

$$\overline{T}_i^{\alpha}(t) = (t^{\alpha} - x_1)(t^{\alpha} - x_2)\cdots(t^{\alpha} - x_i).$$

Hence, the zeros of $\overline{T}_i^{\alpha}(t)$ are

$$t_j = (x_j)^{\frac{1}{\alpha}}, \ j = 1, 2, \cdots, i.$$

□

Lemma 3.3. *$\overline{T}_i^{\alpha}(x)$ is the i-th eigenfunction of the singular Sturm-Liouville problem*

$$\left(x\sqrt{x^{-\alpha}-1}\left(\overline{T}_i^{\alpha}(x)\right)'\right)'+(i\alpha)^2\frac{1}{x\sqrt{x^{-\alpha}-1}}\overline{T}_i^{\alpha}(x)=0. \tag{3.5}$$

Proof. According to definition of the $\overline{T}_i^{\alpha}$, we have

$$\overline{T}_i^{\alpha}(\theta)=\cos(i\theta),\quad \theta=\arccos(2x^{\alpha}-1),\ x\in[0,1].$$

Therefore $\overline{T}_i^{\alpha}$ hold in the following Sturm-Liouville differential equation

$$\frac{d^2}{d\theta^2}\overline{T}_i^{\alpha}(\theta)+i^2\overline{T}_i^{\alpha}(\theta)=0,\ i=0,1,\ldots.$$

Using differential chain rule, we get

$$\left(\overline{T}_i^{\alpha}\right)''\left(\frac{dx}{d\theta}\right)^2+\left(\overline{T}_i^{\alpha}\right)'\left(\frac{dx}{d\theta}\right)\frac{d}{dx}\left(\frac{dx}{d\theta}\right)+i^2\overline{T}_i^{\alpha}=0. \tag{3.6}$$

But

$$\frac{dx}{d\theta}=-\frac{x}{\alpha}\sqrt{x^{-\alpha}-1}, \tag{3.7}$$

Employing Eq. (3.7), Eq. (3.6) can be written as

$$x^2\left(1-x^{\alpha}\right)\left(\overline{T}_i^{\alpha}\right)''+x\left(\frac{2-\alpha}{2}-x^{\alpha}\right)\left(\overline{T}_i^{\alpha}\right)'+(i\alpha)^2x^{\alpha}\overline{T}_i^{\alpha}=0. \tag{3.8}$$

Multiplying both sides of the above equation by $\frac{1}{x}\sqrt{x^{-\alpha}-1}$, we obtain

$$x\sqrt{x^{-\alpha}-1}y''+\sqrt{x^{-\alpha}-1}\left(1-\frac{\alpha}{2(1-x^{\alpha})}\right)y'+(i\alpha)^2\frac{1}{x\sqrt{x^{-\alpha}-1}}y=0.$$

This leads to the desired result. □

3.1. **Function approximation and error estimation.** Let $w_{\alpha}(x)=\frac{1}{x\sqrt{x^{-\alpha}-1}}$ denote a non-negative, integrable, real valued function over the interval $I=(0,1)$. We define

$$L^2_{w_{\alpha}}:=\{v:I\to\mathbb{R}|\ v\text{ is measurable and }\|v\|_{w_{\alpha}}<\infty\},$$

where

$$\|v\|_{w_{\alpha}}=\left(\int_0^1|v(x)|^2w_{\alpha}(x)dx\right)^{\frac{1}{2}},$$

is the norm induced by the inner product

$$\langle u,v\rangle_{w_{\alpha}}=\int_0^1u(x)v(x)w_{\alpha}(x)dx. \tag{3.9}$$

Thus, using Lemma 3.1, $\{\overline{T}_n^{\alpha}(x)\}_{n=0}^{\infty}$ denote a system which are mutually orthogonal under (3.9). For any function $f\in L^2_{w_{\alpha}}$, we approximate

$$f\approx\sum_{k=0}^{N}f_k\overline{T}_k^{\alpha}(x), \tag{3.10}$$

with

$$f_k = \frac{\langle f, \overline{T}_k^\alpha \rangle_{w_\alpha}}{\|\overline{T}_k^\alpha\|^2_{w_\alpha}}. \tag{3.11}$$

The f_k's are the expansion coefficients associated with the family $\{\overline{T}_k^\alpha\}$. We now consider error estimate for approximation using the orthogonal system $\{\overline{T}_n^\alpha(x)\}_{n=0}^\infty$. Let N be any positive integer and

$$V_N^\alpha := \mathrm{span}\{\overline{T}_0^\alpha(x), \overline{T}_1^\alpha(x), \cdots, \overline{T}_N^\alpha(x)\}, \tag{3.12}$$

and, let $P_N^\alpha : L^2_{w_\alpha} \to V_N^\alpha$ be the $L^2_{w_\alpha}$-orthogonal projector defined by

$$\langle P_N^\alpha f - f, \phi \rangle_{w_\alpha} = 0, \qquad \forall \phi \in V_N^\alpha. \tag{3.13}$$

In other hand, $P_N^\alpha f = \sum_{k=0}^N f_k \overline{T}_k^\alpha(x)$ is the best approximation of f out of V_N^α . The following theorem gives the error estimation for this approximation in w_α-norm.

Theorem 3.4. *If $D^{i\alpha} f(x) \in C(0,1]$ for $i = 0, 1, \cdots, N+1$ and $P_N^\alpha f$ be the best approximation to f out of V_N^α, then the error bound can be obtained from the following formula:*

$$\|f - P_N^\alpha f\|_{w_\alpha} \le \frac{M_\alpha}{\Gamma\big((N+1)\alpha + 1\big)} \sqrt{\frac{\sqrt{\pi}}{\alpha} \frac{\Gamma\big(2N + \frac{3}{2}\big)}{\Gamma(2N+3)}},$$

where

$$M_\alpha = \sup\{|D^{N\alpha} f(x)|, \quad x \in (0,1]\}.$$

Proof. Consider the generalized Taylor polynomial:

$$y(x) = \sum_{i=0}^N \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} D^{i\alpha} f(0^+), \; x \in (0,1],$$

in which the following error bound is known:

$$\begin{aligned} |f(x) - y(x)| &= \Big| \frac{x^{(N+1)\alpha}}{\Gamma\big((N+1)\alpha+1\big)} D^{(N+1)\alpha} f(\xi_x) \Big| \\ &\le M_\alpha \frac{x^{(N+1)\alpha}}{\Gamma\big((N+1)\alpha+1\big)}. \end{aligned}$$

Since $P_N^\alpha f(x)$ and $y(x)$ are both in V_N^α and $P_N^\alpha f(x)$ is the best approximation to $f(x)$ out of V_N^α, we get

$$\begin{aligned} \|f - P_N^\alpha f\|^2_{w_\alpha} \le \|f - y\|^2_{w_\alpha} &\le \frac{M_\alpha^2}{\Gamma\big((N+1)\alpha+1\big)^2} \int_0^1 \frac{x^{2(N+1)\alpha}}{x\sqrt{x^{-\alpha} - 1}} dx \\ &= \frac{M_\alpha^2}{\Gamma\big((N+1)\alpha+1\big)^2} \frac{2}{\alpha} \int_0^\infty \Big(\frac{1}{1+u^2}\Big)^{2N+3} du \\ &= \frac{M_\alpha^2}{\Gamma\big((N+1)\alpha+1\big)^2} \frac{2}{\alpha} \frac{\sqrt{\pi}}{2} \frac{\Gamma\big(2N+\frac{3}{2}\big)}{\Gamma(2N+3)}, \end{aligned}$$

this leads to the desired result. □

4. The operational matrix of fractional derivative

Let us introduce some notation.

$$\begin{aligned} &T_\alpha(x) = [\overline{T}_0^\alpha(x), \overline{T}_1^\alpha(x), \cdots, \overline{T}_N^\alpha(x)]^T, \\ &X_\alpha(x) = [1, x^\alpha, x^{2\alpha}, \cdots, x^{N\alpha}]^T. \end{aligned} \tag{4.1}$$

By using Eq. (2.7) we can write:

$$T_\alpha(x) = \mathbf{F} X_\alpha, \tag{4.2}$$

or

$$\overline{T}_i^\alpha(x) = \sum_{j=0}^{N} f_{ij} x^{j\alpha}, \quad i = 0, 1, 2, \cdots, N. \tag{4.3}$$

The fractional derivative of order γ of the vector $T_\alpha(x)$ can be expressed by

$$D^\gamma T_\alpha(x) \simeq \mathbf{D}^{(\gamma)} T_\alpha(x),$$

where $\mathbf{D}^{(\gamma)}$ is the $(N+1) \times (N+1)$ operational matrix of the fractional derivative. In this section, our aim is to construct $\mathbf{D}^{(\gamma)}$.

Remark 4.1. It is important to mention here that, in this paper we assume $\alpha \in \mathbb{N}_0$ or $\alpha > \lfloor \gamma \rfloor$ when $\alpha \notin \mathbb{N}$.

Lemma 4.2. *Let*

$$k = \begin{cases} \text{the largest integer such that } \; k\alpha < \lceil \gamma \rceil, & \alpha \in \mathbb{N}_0, \\ 0, & \alpha \notin \mathbb{N} \; \text{ and } \alpha > \lfloor \gamma \rfloor. \end{cases} \tag{4.4}$$

Then, the Caputo fractional derivative of order γ of the vector $X_\alpha(x)$ can be expressed as

$$D^\gamma X_\alpha(x) = \overline{\boldsymbol{D}}_\gamma X_\alpha^\gamma(x), \tag{4.5}$$

where $\overline{\boldsymbol{D}}_\gamma$ is a $(n+1) \times (n+1)$ diagonal matrix

$$\overline{\boldsymbol{D}}_\gamma = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\gamma)} & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\gamma)} \end{bmatrix}, \tag{4.6}$$

and

$$X_\alpha^\gamma(x) = \left[0, \cdots, 0, x^{(k+1)\alpha-\gamma}, x^{(k+2)\alpha-\gamma}, \cdots, x^{N\alpha-\gamma}\right]^T. \tag{4.7}$$

Proof. Using Eq. (2.11) we have

$$D^{\gamma}X_{\alpha}(x) = \begin{bmatrix} 0 \\ D^{\gamma}x^{\alpha} \\ \vdots \\ D^{\gamma}x^{N\alpha} \end{bmatrix} = \overline{\mathbf{D}}_{\gamma}X_{\alpha}^{\gamma}(x),$$

where

$$\overline{\mathbf{D}}_{\gamma} = \begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\Gamma((k+1)\alpha+1)}{\Gamma((k+1)\alpha+1-\gamma)} & \cdots & 0 \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma(N\alpha+1-\gamma)} \end{bmatrix},$$

and

$$X_{\alpha}^{\gamma}(x) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ x^{(k+1)\alpha-\gamma} \\ \vdots \\ x^{N\alpha-\gamma} \end{bmatrix}.$$

□

It is important to note that, in $\overline{\mathbf{D}}_{\gamma}$, the first $k+1$ rows, are all zero.

Lemma 4.3. *With the notations of Lemma 4.2. We can approximate $X_{\alpha}^{\gamma}(x)$ as:*

$$X_{\alpha}^{\gamma}(x) \simeq \boldsymbol{B}T_{\alpha}(x), \tag{4.8}$$

where $\boldsymbol{B} = (b_{ij})$ is a $(N+1)\times(N+1)$ matrix with the following entries:

$$b_{ij} = \begin{cases} 0, & \begin{cases} i = 0,1,2,\cdots,k, \\ j = 0,1,2,...,N, \end{cases} \\ \frac{\sqrt{\pi}}{h_j}\sum_{l=0}^{j-1} f_{jl}\frac{\Gamma\left(i+l-\frac{\gamma}{\alpha}+\frac{1}{2}\right)}{\Gamma\left(i+l-\frac{\gamma}{\alpha}+1\right)}, & \begin{cases} i = k+1,k+2,\cdots,N, \\ j = 0,1,2,...,N. \end{cases} \end{cases}$$

Proof. Clearly, for $i = 0,1,\cdots,k$, we have $b_{ij} = 0$. Now, for some $i > k$, approximate $x^{i\alpha-\gamma}$ by $N+1$ terms of fractional-order Chebyshev series, we get

$$x^{i\alpha-\gamma} \simeq \sum_{j=0}^{N} b_{ij}\overline{T}_{j}^{\alpha}(x).$$

By using Eqs. (3.11) and (4.3) we obtained

$$\begin{aligned}
b_{ij} &= \frac{\alpha}{h_j}\int_0^1 x^{i\alpha-\gamma}\overline{T}_j^{\alpha}(x)\frac{1}{x\sqrt{x^{-\alpha}-1}}dx\\
&= \frac{\alpha}{h_j}\int_0^1 x^{i\alpha-\gamma}\sum_{l=0}^{j-1} f_{jl}x^{l\alpha}\frac{1}{x\sqrt{x^{-\alpha}-1}}dx\\
&= \frac{\alpha}{h_j}\sum_{l=0}^{j-1} f_{jl}\int_0^1 x^{i\alpha-\gamma}x^{l\alpha}\frac{1}{x\sqrt{x^{-\alpha}-1}}dx\\
&= \frac{\alpha}{h_j}\sum_{l=0}^{j-1} f_{jl}\int_0^1 G(x,i,l,\alpha,\gamma)dx.
\end{aligned}$$

Set $u=\sqrt{x^{-\alpha}-1}$, we get

$$\begin{aligned}
\int_0^1 G(x,i,l,\alpha,\gamma)dx &= \frac{2}{\alpha}\int_0^{\infty}\frac{du}{(u^2+1)^{i-\frac{\gamma}{\alpha}+l}}\\
&= \frac{2}{\alpha}\frac{\sqrt{\pi}}{2}\frac{\Gamma(i+l-\frac{\gamma}{\alpha}+\frac{1}{2})}{\Gamma(i+l-\frac{\gamma}{\alpha}+1)}.
\end{aligned}$$

Thus

$$b_{ij}=\frac{\sqrt{\pi}}{h_j}\sum_{l=0}^{j-1} f_{jl}\frac{\Gamma(i+l-\frac{\gamma}{\alpha}+\frac{1}{2})}{\Gamma(i+l-\frac{\gamma}{\alpha}+1)},$$

where h_j is defined in (2.5). □

We are now ready to state the main result of this section.

Theorem 4.4. *Let $T_\alpha(x)$ be FCFs vector, $\boldsymbol{D}^{(\gamma)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional derivative of order $\gamma>0$ in Caputo sense and $\alpha\in\mathbb{N}_0$ or $\alpha>\lfloor\gamma\rfloor$ when $\alpha\notin\mathbb{N}$ then*

$$\boldsymbol{D}^{(\gamma)}\simeq \boldsymbol{F}\,\overline{\boldsymbol{D}}_\gamma\,\boldsymbol{B},$$

where $\overline{\boldsymbol{D}}_\gamma, \boldsymbol{B}$ and $\boldsymbol{F}$ are given in Eqs. (4.6), (4.8) and (2.7) respectively.

Proof. Applying Eq. (4.2), Lemmas 4.2 and 4.3, we can write γth order fractional derivative of $T_\alpha(x)$ as

$$\begin{aligned}
D^{\gamma}T_\alpha(x) &= \mathbf{F}D^{\gamma}X_\alpha(x)=\mathbf{F}\overline{\mathbf{D}}_\gamma X_\alpha^{\gamma}(x)\\
&\simeq \mathbf{F}\overline{\mathbf{D}}_\gamma\mathbf{B}T_\alpha(x)\\
&= \mathbf{D}^{(\gamma)}T_\alpha(x).
\end{aligned}$$

□

Proposition 4.5. *In particular, when $\gamma=\alpha$, $\boldsymbol{D}^{(\alpha)}$ can be computed as*

$$\boldsymbol{D}^{(\alpha)}=\boldsymbol{F}\,\overline{\overline{\boldsymbol{D}}}_\alpha\boldsymbol{F}^{-1},$$

where $\overline{\overline{\boldsymbol{D}}}_\alpha$ *is the* $(N+1)\times(N+1)$ *matrix given by*

$$\overline{\overline{\boldsymbol{D}}}_\alpha = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \frac{\Gamma(\alpha+1)}{\Gamma(1)} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{\Gamma(N\alpha+1)}{\Gamma\left((N-1)\alpha+1\right)} & 0 \end{bmatrix}. \tag{4.9}$$

Proof. Using (2.11) we have

$$D^\alpha x^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(1)}, \quad D^\alpha x^{2\alpha} = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} x^\alpha, \; \ldots,$$
$$D^\alpha x^{N\alpha} = \frac{\Gamma(N\alpha+1)}{\Gamma\big((N-1)\alpha+1\big)} x^{(N-1)\alpha}. \tag{4.10}$$

Thus, αth order fractional derivative of $X_\alpha(x)$ is written in the matrix form

$$D^\alpha X_\alpha(x) = \overline{\overline{\mathbf{D}}}_\alpha X_\alpha(x), \tag{4.11}$$

According to Eq. (4.2), we have

$$X_\alpha(x) = \mathbf{F}^{-1} T_\alpha(x), \tag{4.12}$$

Therefore, using Eqs. (4.2), (4.11) and (4.12) we obtain

$$\begin{aligned} D^\alpha T_\alpha(x) &= \mathbf{F}\, D^\alpha X_\alpha(x) = \mathbf{F}\, \overline{\overline{\mathbf{D}}}_\alpha X_\alpha(x) \\ &= \mathbf{F}\, \overline{\overline{\mathbf{D}}}_\alpha \mathbf{F}^{-1} T_\alpha(x). \end{aligned}$$

□

Remark 4.6. It should be noted that, the operational matrix of the fractional derivative of $T_\alpha(x)$ with $\alpha = 1$, is in complete agreement with Chebyshev operational matrix of fractional derivative obtained by Doha et al. (see [14] Eq. (3.5)).

5. Applications of the operational matrix of fractional derivatives

In this section, in order to show the efficiency of operational matrix of fractional derivative, we apply it to solve multi-order fractional differential equation.

5.1. **Linear multi-order FDEs.** Consider the linear multi-order FDE

$$D^\gamma y(x) = a_1 D^{\beta_1} y(x) + \cdots + a_k D^{\beta_k} y(x) + a_{k+1} y(x) + a_{k+2} g(x), \tag{5.1}$$

with initial conditions

$$y^{(i)}(0) = d_i, \qquad i = 0, \ldots m, \tag{5.2}$$

where a_j, for $j = 1, ..., k+2$ are real constant coefficients and also $m < \gamma \leq m+1$, $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \gamma$. To solve problem (5.1) and (5.2), suppose $y(x)$ and $g(x)$ be in $L^2_{w_\alpha}$. Using the $L^2_{w_\alpha}$-orthogonal projector defined in (3.13), we approximate $y(x)$ and $g(x)$ by FCFs as

$$y(x) \simeq \sum_{i=0}^{N} c_i \overline{T}_i^{\alpha}(x) = C^T T_\alpha(x), \tag{5.3}$$

$$g(x) \simeq \sum_{i=0}^{N} g_i \overline{T}_i^{\alpha}(x) = G^T T_\alpha(x), \tag{5.4}$$

where vector $G = [g_0, ..., g_N]^T$ is known but $C = [c_0, ..., c_N]^T$ is an unknown vector. By using Theorem 4.4 and Eq. (5.3), we have

$$D^\gamma y(x) \simeq C^T D^\gamma T_\alpha(x) \simeq C^T \mathbf{D}^{(\gamma)} T_\alpha(x), \tag{5.5}$$

$$D^{\beta_j} y(x) \simeq C^T D^{\beta_j} T_\alpha(x) \simeq C^T \mathbf{D}^{(\beta_j)} T_\alpha(x), \qquad j = 1, \dots k. \tag{5.6}$$

Employing Eqs. (5.3), (5.4), (5.5) and (5.6) the residual $R_N(x)$ for Eq. (5.1) can be written as

$$R_N(x) \simeq \left(C^T \mathbf{D}^{(\gamma)} - C^T \sum_{j=1}^{k} a_j \mathbf{D}^{(\beta_j)} - a_{k+1} C^T - a_{k+2} G^T \right) T_\alpha(x). \tag{5.7}$$

As in a typical tau method [6], we generate $m - n$ linear equations by applying

$$\langle R_N(x), \overline{T}_j^{\alpha}(x) \rangle_{w_\alpha} = \int_0^1 R_N(x) \overline{T}_j^{\alpha}(x) w_\alpha(x) dx = 0, \quad j = 0, 1, ..., N - m - 1. \tag{5.8}$$

Also, by using Theorem 4.4 and by substituting Eq. (5.3) in Eq. (5.2) we get

$$y^{(i)}(0) = C^T \mathbf{D}^{(i)} T_\alpha(0) = d_i, \quad i = 0, 1, ..., m. \tag{5.9}$$

Equations (5.8) and (5.9) generate $(N - m)$ and $(m + 1)$ set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $y(x)$ given in Eq. (5.3) can be calculated.

5.2. **Nonlinear multi-order FDFs.** Consider the nonlinear multi-order FDE

$$F\left(x, y(x), D^{\beta_1} y(x), \dots, D^{\beta_k} y(x)\right) = 0, \tag{5.10}$$

with boundary or supplementary conditions

$$H_i(y(\xi_i), y'(\xi_i), ..., y^{(p)}(\xi_i)) = d_i, \qquad i = 0, 1, ..., p, \tag{5.11}$$

where $0 \le p < \max\{\beta_i,\ i = 1, ..., k\} \le p + 1,\ \xi_i \in [0, 1], i = 0, ..., p$ and H_i are linear combinations of $y(\xi_i), y'(\xi_i), ..., y^{(p)}(\xi_i)$. In order to use FCFs for this problem, we approximate $y(x)$ by Eq. (5.3). Now, using Theorem 4.4 we have

$$D^{\beta_j} y(x) \simeq C^T D^{\beta_j} T_\alpha(x) \simeq C^T \mathbf{D}^{(\beta_j)} T_\alpha(x), \qquad j = 1, \dots k. \tag{5.12}$$

By substituting these equations in Eq. (5.10) we get

$$F\left(x, C^T T_\alpha(x), C^T \mathbf{D}^{(\beta_1)} T_\alpha(x), \dots, C^T \mathbf{D}^{(\beta_k)} T_\alpha(x)\right) = 0. \tag{5.13}$$

Similarly, substituting Eq. (5.3) in Eq. (5.11) yields

$$H_i(C^T T_\alpha(\xi_i), C^T \mathbf{D}^{(1)} T_\alpha(\xi_i), ..., C^T \mathbf{D}^{(p)} T_\alpha(\xi_i)) = d_i, \qquad i = 0, 1, ..., p. \tag{5.14}$$

To find the solution $y(x)$, we first collocate Eq. (5.13) at $(N-p)$ points. For suitable collocation points we use the roots of $\overline{T}^{\alpha}_{N-p}(x)$ which computed in (3.3). These equations together with Eq. (5.14) generate $(N+1)$ algebraic equations which can be solved to find $c_i, \;\; i = 0, ..., N$. Consequently the unknown function $y(x)$ given in Eq. (5.3) can be calculated.

6. Numerical examples

In this section, we exemplify the proposed approximation procedure with some examples.

Example 1. We start with the Bagley-Torvik equation

$$AD^2 y(x) + BD^{\frac{3}{2}} y(x) + Cy(x) = g(x).$$

This equation arises, for example, in the modelling of the motion of a rigid plate immersed in a Newtonian fluid [42]. Here $A = M$ is the mass of thin rigid plate and $C = K$ is the stiffness of the spring. Also, $B = 2S\sqrt{\mu\rho}$ where S is area of plate immersed in Newtonian fluid, μ is viscosity, and ρ is the fluid density. This equation has attracted much attention and has been studied by many authors [14, 22, 39]. Here, we consider inhomogeneous Bagley-Torvik equation

$$D^2 y(x) + D^{\frac{3}{2}} y(x) + y(x) = g(x),$$

in two cases.

Case a: $g(x) = 1 + x, \; y(0) = 1, \; y'(0) = 1.$

The exact solution of the problem in this case is $y(x) = 1 + x$. By applying the technique described in Section 5.1 with $N = 2$ and $\alpha = 1$, we approximate solution as

$$y(x) = c_0 \overline{T}^1_0(x) + c_1 \overline{T}^1_1(x) + c_2 \overline{T}^1_2(x) = C^T T_1(x).$$

Here, we have

$$\mathbf{D}^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}, \qquad \mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 16 & 0 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(\frac{3}{2})} = \left(\frac{1}{\sqrt{\pi^3}}\right) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 64 & \frac{128}{3} & -\frac{128}{15} \end{pmatrix}, \qquad G = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}.$$

Therefore using Eq. (5.8) we obtain

$$c_0 + \left(16 + 64\pi^{-\frac{3}{2}}\right) c_2 = \frac{3}{2}. \tag{6.1}$$

Now, by applying Eq. (5.9) we have

$$C^T T_1(0) = c_0 - c_1 + c_2 = 1, \tag{6.2}$$

$$C^T \mathbf{D}^{(1)} T_1(0) = 2c_1 - 8c_2 = 1. \tag{6.3}$$

Finally by solving linear system of three equations, (6.1), (6.2) and (6.3) we get

$$c_0 = \frac{3}{2}, \quad c_1 = \frac{1}{2}, \quad c_2 = 0.$$

Thus

$$y(x) = \left(\frac{3}{2}, \frac{1}{2}, 0\right) \begin{pmatrix} 1 \\ 2x - 1 \\ 8x^2 - 8x + 1 \end{pmatrix} = 1 + x,$$

which is the exact solution.

Case b: $g(x) = x^2 + 2 + 4\sqrt{\frac{x}{\pi}},\ y(0) = 0,\ y(1) = 1.$

In this case, the problem has the exact solution $y(x) = x^2$. Taking $\alpha = 2$ and $N = 2$, we obtain:

$$\mathbf{D}^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 32 & 48 & 0 \end{pmatrix},$$

$$\mathbf{D}^{(\frac{3}{2})} = \begin{pmatrix} 0 & 0 & 0 \\ 16\frac{\sqrt{2}\Gamma(\frac{3}{4})^2}{\pi^2} & \frac{32}{5}\frac{\sqrt{2}\Gamma(\frac{3}{4})^2}{\pi^2} & -\frac{32}{15}\frac{\sqrt{2}\Gamma(\frac{3}{4})^2}{\pi^2} \\ \frac{1472}{25}\frac{\sqrt{2}\Gamma(\frac{3}{4})^2}{\pi^2} & \frac{1664}{15}\frac{\sqrt{2}\Gamma(\frac{3}{4})^2}{\pi^2} & \frac{3712}{195}\frac{\sqrt{2}\Gamma(\frac{3}{4})^2}{\pi^2} \end{pmatrix}.$$

After solving the system of 3 algebraic equations the unknown coefficients are computed as:

$$c_0 = \frac{1}{2}, \quad c_1 = \frac{1}{2}, \quad c_2 = 0.$$

Thus

$$y(x) = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \begin{pmatrix} 1 \\ 2x^2 - 1 \\ 8x^4 - 8x^2 + 1 \end{pmatrix} = x^2,$$

which is the exact solution. **Example 2.** As the second example, consider the fractional differential equation [24]

$$D^{0.5} y(x) + y(x) = \sqrt{x} + \frac{\sqrt{\pi}}{2},$$
$$y(0) = 0.$$

Lakestani et al. [24] applied a method based on linear cardinal B-spline functions for the approximate solution of this example. Regarding example 1, in [24], the best result is achieved by solving a system of equations with 2^8+1 set of algebraic equations and the maximum absolute error is 7.8×10^{-5}. We solved this problem, by applying the technique described in Section 5.1, with $\alpha = \frac{1}{2}$ and $N = 1$. Here, we have

$$\mathbf{D}^{(\frac{1}{2})} = \begin{pmatrix} 0 & 0 \\ \sqrt{\pi} & 0 \end{pmatrix}, G = \begin{pmatrix} \frac{1+\sqrt{\pi}}{2} \\ \\ \frac{1}{2} \end{pmatrix}.$$

Therefore, we obtain the following system of algebraic equations:

$$\begin{cases} c_0 - c_1 = 0, \\ c_0 + \sqrt{\pi} c_1 = \frac{1}{2}(1+\sqrt{\pi}). \end{cases}$$

Solving these equations yields

$$c_0 = \frac{1}{2}, c_1 = \frac{1}{2}.$$

Thus

$$y(x) = \left(\frac{1}{2}, \frac{1}{2}\right) \begin{pmatrix} 1 \\ 2x^{\frac{1}{2}} - 1 \end{pmatrix} = \sqrt{x},$$

which is the exact solution.

Example 3. Consider the following linear initial value problem [18, 36]

$$\begin{aligned} &D^{\gamma} y(x) + y(x) = 0, \\ &y(0) = 1, \quad 0 < \gamma \leq 1. \end{aligned}$$

The exact solution for this problem is given by $y(x) = E_{\gamma}(-x^{\gamma})$, where $E_{\gamma}(t)$ is the Mittag-Leffler function defined by [12]

$$E_{\gamma}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\gamma + 1)}, \quad \gamma > 0,$$

We solve this problem with placing $\alpha = \gamma$. Here, we use the least square norm as

$$e_N = \sqrt{\sum_{i=0}^{N} \left(\tilde{y}_N(t_i) - y(t_i)\right)^2},$$

where $\tilde{y}_N$ is the approximated solution and t_i are given in Eq. (3.3). In Figure 1, the approximation error e_N as a function of the discretization parameter N for different values of γ is plotted in a semi-log coordinate system. As one can see, e_N decreases rapidly. Also Figure 1 shows that our new approximate solution is in good agreement with the collocation method based on Müntz polynomials given in [18]. Furthermore the graph of absolute error function $|y(x) - \tilde{y}_{10}(x)|$ for $\gamma = 0.2, 0.4, 0.6$ and 0.8 is given in Figure 2. From Figure 2, we can see that the new method presented in the current paper provides accurate results even by using $N = 10$.

FIGURE 1. Approximation error e_N for various values of γ for Example 3.

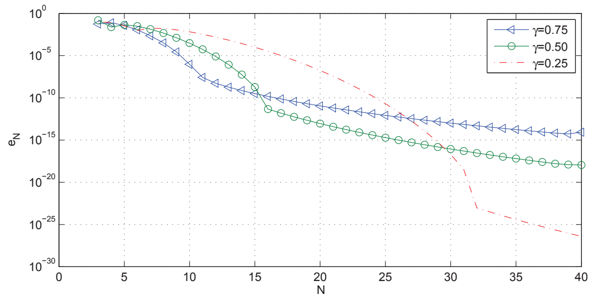
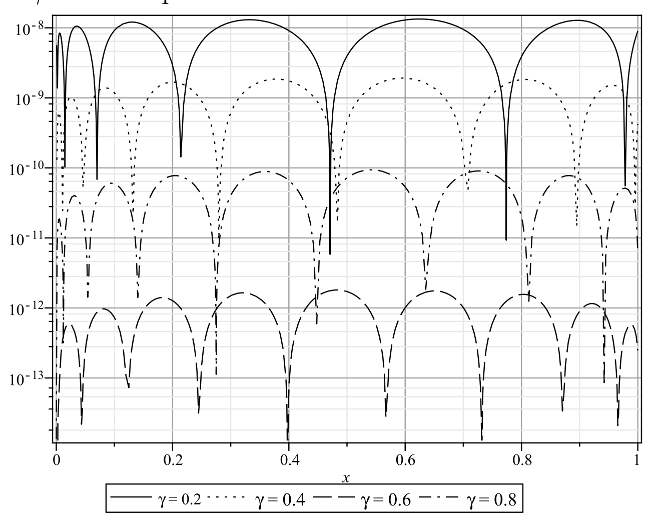


FIGURE 2. Plot of absolute error function $|y(x) - \tilde{y}_{10}(x)|$ for various values of γ for Example 3.

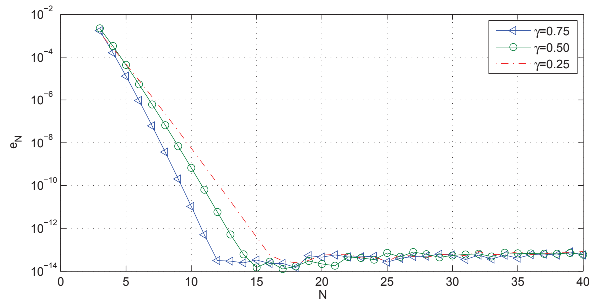


Example 4. Let us consider the following nonlinear fractional initial value problem [12, 18, 39]

$$
\begin{aligned}
D^{\gamma}y(x) =& \frac{40320}{\Gamma(9-\gamma)}x^{8-\gamma} - 3\frac{\Gamma(5+\frac{\gamma}{2})}{\Gamma(5-\frac{\gamma}{2})}x^{4-\frac{\gamma}{2}} \\
&+ \frac{9}{4}\Gamma(\gamma+1) + \left(\frac{3}{2}x^{\frac{\gamma}{2}} - x^4\right)^3 - y(x)^{\frac{3}{2}}, \quad \gamma, x \in (0,1), \\
y(0) =& 0.
\end{aligned}
$$

The exact solution is $y(x) = x^8 - 3x^{4+\frac{\gamma}{2}} + \frac{9}{4}x^{\gamma}$. The approximation error e_N for $\alpha = \gamma$ and for different values of γ is plotted in Figure 3. As expected, the error e_N

FIGURE 3. Approximation error e_N for various values of γ for Example 4.



shows an exponential decay. Also Table 1, shows the absolute errors obtained by the present method for $\alpha = \gamma$ and for different values of γ. These results demonstrate the spectral accuracy of the present method for this example.

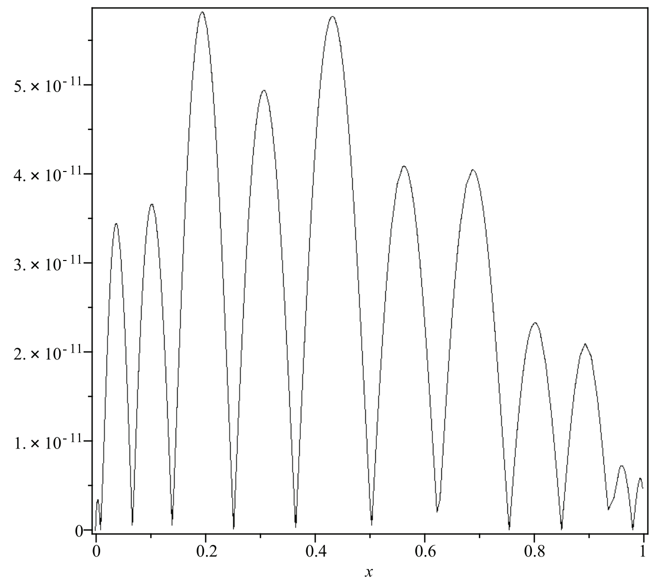
TABLE 1. Absolute values of errors for Example 3.

		x_i				
γ	N	0.125	0.375	0.5	0.625	0.875
0.25	10	4.7×10^{-4}	6.1×10^{-3}	4.4×10^{-3}	8.8×10^{-3}	1.2×10^{-2}
	15	3.2×10^{-5}	6.6×10^{-5}	1.3×10^{-4}	9.1×10^{-5}	1.6×10^{-4}
	20	1.2×10^{-7}	1.4×10^{-7}	1.4×10^{-7}	8.5×10^{-8}	8.2×10^{-8}
	25	1.5×10^{-11}	1.7×10^{-11}	7.0×10^{-12}	2.3×10^{-12}	2.6×10^{-11}
0.50	10	9.2×10^{-5}	3.1×10^{-5}	1.7×10^{-4}	2.0×10^{-4}	2.4×10^{-4}
	15	6.8×10^{-10}	3.2×10^{-10}	4.9×10^{-10}	7.7×10^{-10}	5.7×10^{-10}
	20	3.9×10^{-14}	2.8×10^{-14}	3.2×10^{-15}	2.7×10^{-5}	3.1×10^{-5}
	25	7.4×10^{-16}	6.7×10^{-16}	4.0×10^{-16}	4.7×10^{-16}	2.9×10^{-16}
0.75	10	1.8×10^{-7}	6.7×10^{-8}	2.9×10^{-7}	4.8×10^{-7}	1.6×10^{-7}
	15	1.0×10^{-10}	1.1×10^{-10}	9.9×10^{-11}	8.2×10^{-11}	2.4×10^{-11}
	20	2.2×10^{-12}	1.1×10^{-12}	2.5×10^{-12}	1.3×10^{-12}	6.1×10^{-13}
	25	2.0×10^{-13}	1.5×10^{-13}	7.8×10^{-14}	1.4×10^{-14}	3.6×10^{-14}
1.0	10	9.8×10^{-8}	4.7×10^{-8}	4.3×10^{-8}	1.0×10^{-7}	8.1×10^{-8}
	15	5.7×10^{-9}	1.1×10^{-9}	3.3×10^{-9}	7.7×10^{-10}	1.6×10^{-9}
	20	1.5×10^{-10}	7.3×10^{-11}	6.9×10^{-11}	1.6×10^{-10}	9.0×10^{-11}
	25	3.7×10^{-11}	7.1×10^{-12}	5.3×10^{-12}	3.9×10^{-12}	9.0×10^{-12}

Example 5. As the last example, consider the following fractional Riccati equation [22, 33]

$$D^\gamma y(x) = -y^2(x) + 1, \quad x > 0$$
$$y(0) = 0, \quad \gamma \in (0, 1].$$

FIGURE 4. Plot of the absolute error function $|y(x) - \tilde{y}_{12}(x)|$ with $\gamma = \alpha = 1$ for Example 5.



The exact solution, when $\gamma = 1$ is $y(x) = (e^{2x}+1)/(e^{2x}-1)$. We solved this problem, by applying the technique described in Section 5.2. In Figure 4, we plot the absolute error function $|y(x) - \tilde{y}_{12}(x)|$ for $\gamma = \alpha = 1$, which shows that the new numerical solution is closely correlated to the exact solution. The exact solutions for the values of $\gamma \neq 1$ are not exist. Therefore, see [22], to show efficient of our method we define the norm of residual error as follows

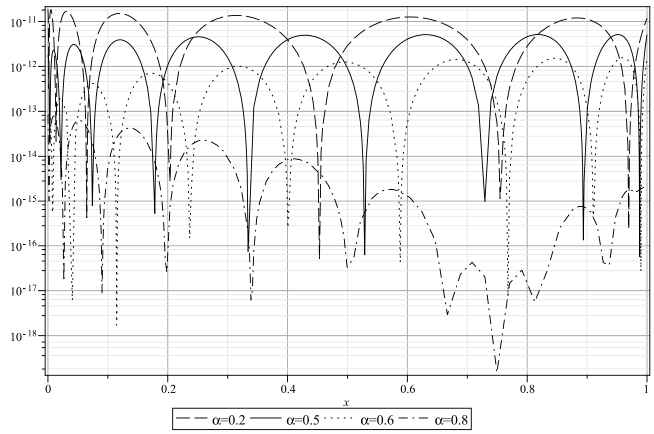
$$\mathrm{Res}_N(x) = D^{\gamma}\tilde{y}_N(x) + \tilde{y}_N^2(x) - 1, \qquad \|\mathrm{Res}_N(x)\|^2 = \int_0^1 \mathrm{Res}_N^2(x)dx \tag{6.4}$$

In Figure 5 we plot $\|\mathrm{Res}_N(x)\|^2$ obtained by the present method with $N = 10$ and various values of $\gamma = \alpha$. Figures 4 and 5, demonstrate the advantages and the accuracy of the present method for solving nonlinear fractional Riccati equation.

7. CONCLUSION

In this paper, we first introduce fractional-order Chebyshev functions, then we obtain a new fractional derivative operational matrix for these orthogonal functions. This matrix can be used to solve fractional differential equations, like that of the other orthogonal functions. The method is general, easy to implement, and yields very accurate results. Illustrative examples are included to demonstrate the validity and applicability of the technique and the exact solutions are obtained for some test problems.

FIGURE 5. Plot of the $\|\mathrm{Res}_N(x)\|^2$ for Example 5, with $N = 10$ and various values of $\gamma = \alpha$.



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