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Asymptotic analysis of the Wright function with a large parameter

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ABSTRACT

In this paper, using the exponential conformal map for the Hankel contour we show a new Schläfli-type integral representation for the Wright function. We apply the steepest descent method and the Lagrange expansion to find the asymptotic expansions of Wright function for the large parameter. We study two cases for the stationary points and discuss the associated asymptotic expansions. The results extend the asymptotic expansions of the Bessel functions of the first and second kinds.

1. Introduction and preliminaries

1.1. Definitions and notations

The Wright function introduced by Edward Maitland Wright [38–40]

$$W(c,d;z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(cn+d)},\quad c>-1,\ d\in\mathbb{C},\ z\in\mathbb{C}, \tag{1.1}$$

is a particular case of the generalized hypergeometric functions. This function along with the Mittag-Leffler function [14] are two important special functions and have pivotal roles in the theory of fractional calculus and associated structures in the applied mathematics, for example see [1–8,11–13,15–18,21,23–26,36,37]. The problem of the asymptotic behavior of the Wright function for large values of z was stated by Wright [39–41] and this property and more geometric properties were developed by other researchers [9,19,20,22, 28–30,32]. The interested reader is referred to the cited references in the historical and bibliographical notes

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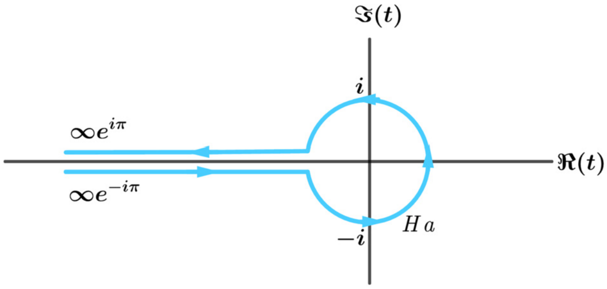


Fig. 1. The Hankel contour for Wright function $W(c,d;z)$.

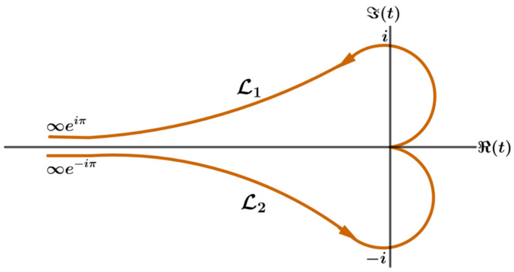


Fig. 2. The associated contours for the functions $H^{(1)}(c,d;z)$ and $H^{(2)}(c,d;z)$.

in [14, Chapter 7]. An alternative representation for the Wright function is considered in the literature by the following integral representation [31]

$$W(c,d;z) = \frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-d} e^{t+zt^{-c}}\,dt, \quad c > -1,\ d \in \mathbb{C},\ z \in \mathbb{C}, \tag{1.2}$$

where Ha is the Hankel contour presented in Fig. 1. Here, we set the modified Wright function

$$\mathscr{W}(c,d;z) = \frac{z^{d-1}}{2^{d-1}} W(c,d;-\frac{z^{c+1}}{2^{c+1}}) = \frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-d} e^{\frac{z}{2}(t-t^{-c})}\,dt, \quad |\arg(z)| < \frac{\pi}{2}, \tag{1.3}$$

as generalization of the Bessel function of first kind [35]

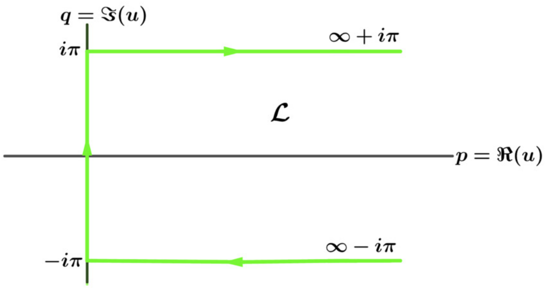
$$J_\nu(z) = \frac{z^\nu}{2^\nu} W(1,\nu+1;-\frac{z^2}{4}) = \frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-\nu-1} e^{\frac{z}{2}(t-t^{-1})}\,dt, \quad |\arg(z)| < \frac{\pi}{2}, \tag{1.4}$$

and intend to use the steepest descent method and get the asymptotic expansion of Wright function (1.3) for the large parameter d and the large value z, simultaneously. Motivated by the asymptotic expansion of the Bessel function for large parameter ν [35], we define two modifications of the Wright function denoted by $H^{(1)}(c,d;z)$ and $H^{(2)}(c,d;z)$ in the following forms

$$H^{(j)}(c,d;z) = \frac{1}{i\pi}\int_{\mathscr{L}_j} t^{-d} e^{\frac{z}{2}(t-t^{-c})}\,dt, \quad c > -1,\ d \in \mathbb{C},\ |\arg(z)| < \frac{\pi}{2},\ j = 1,2, \tag{1.5}$$

where $\mathscr{L}_1$ and $\mathscr{L}_2$ are shown in Fig. 2. It is obvious that the Wright function can be interpreted as

$$\mathscr{W}(c,d;z) = \frac{1}{2}\left[H^{(1)}(c,d;z) + H^{(2)}(c,d;z)\right]. \tag{1.6}$$



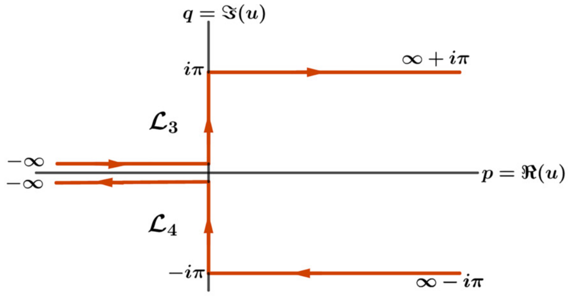


Fig. 3. The mapped contours corresponding to (1.7) (left) and (1.8) (right), respectively.

We apply the map $t = e^u$ and set $\nu = d - 1$ in (1.2) and (1.5) to get new integral representations as

$$\mathscr{W}(c, \nu + 1; z) = \frac{1}{2\pi i} \int\limits_{\mathscr{L}} e^{ze^{\frac{1-c}{2}u} \sinh(\frac{1+c}{2}u) - \nu u} du, \quad c > -1,\ \nu \in \mathbb{C},\ |\arg(z)| < \frac{\pi}{2}, \tag{1.7}$$

$$H^{(j)}(c, \nu + 1; z) = \frac{1}{\pi i} \int\limits_{\mathscr{L}_{j+2}} e^{ze^{\frac{1-c}{2}u} \sinh(\frac{1+c}{2}u) - \nu u} du, \quad c > -1,\ \nu \in \mathbb{C},\ |\arg(z)| < \frac{\pi}{2},\ j = 1, 2, \tag{1.8}$$

where $\mathscr{L}$, $\mathscr{L}_3$ and $\mathscr{L}_4$ are the mapped contours in the u-plane shown in Fig. 3. The integral (1.7) is a generalization of the Schläfli integral which has been stated in the literature [35].

1.2. Steepest descent method

Theorem 1.1. *[10,27,33] Let*

$$I(\nu) = \int\limits_C g(u) e^{\nu h(u)} du, \tag{1.9}$$

be an integral with the analytic functions g and h in a region including C, such that

$$\frac{d^r h(u)}{du^r}\Big|_{u=u_0} = 0, \quad r = 1, 2, \ldots, n-1, \quad \frac{d^n h(u)}{du^n}\Big|_{u=u_0} = ae^{i\alpha}, \quad a > 0, \tag{1.10}$$

and $u - u_0 = \rho e^{i\theta}$. Then, the directions of steepest descent at the stationary point $u = u_0$ (the roots of equation $h'(u) = 0$) are given by

$$\theta_r = -\frac{\alpha}{n} + (2r + 1)\frac{\pi}{n}, \quad r = 0, 1, \ldots, n-1, \tag{1.11}$$

and the steepest descent curves are obtained by $\Im(h(u)) = \Im(h(u_0))$. Moreover, the following leading asymptotic term holds for $I(\nu)$

$$I(\nu) \sim \frac{g_0}{n} \Big(\frac{n!}{\nu |h^{(n)}(u_0)|} \Big)^{\frac{\beta}{n}} \Gamma(\frac{\beta}{n}) e^{\nu h(u_0) + i\beta\theta_r}, \quad r = 0, 1, \cdots, n-1, \tag{1.12}$$

where for small "o" we have

$$g(u) = g_0(u - u_0)^{\beta - 1} + o(u - u_0)^{\beta - 1}, \quad \Re(\beta) > 0. \tag{1.13}$$

Theorem 1.2. *(Lagrange's Theorem [34]) Let z be the root of $z = a + ug(z)$ which has the value $z = a$ when $u = 0$. Also, $g(z)$ is analytic inside and on a circle C containing $z = a$. Then, for the analytic function $f(z)$ inside and on C, we have*

$$f(z) = f(a) + \sum_{n=1}^{\infty} \frac{u^n}{n!} \frac{d^{n-1}}{da^{n-1}} \{f'(a)[g(a)]^n\}. \tag{1.14}$$

In order to employ the steepest descent method for the integral (1.3), at first step we set

$$z = x = \nu\lambda, \quad x, \nu \in \mathbb{R}, \tag{1.15}$$

$$h(u) = \lambda e^{\frac{1-c}{2}u} \sinh(\frac{1+c}{2}u) - u, \tag{1.16}$$

and represent the desired functions as

$$\mathscr{W}(c, \nu+1; \nu\lambda) = \frac{1}{2\pi i} \int_{\mathscr{L}} e^{\nu h(u)} du = \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} e^{\nu h(u)} du, \tag{1.17}$$

$$H^{(1)}(c, \nu+1; \nu\lambda) = \frac{1}{2\pi i} \int_{\mathscr{L}_3} e^{\nu h(u)} du = \frac{1}{2\pi i} \int_{-\infty}^{\infty + i\pi} e^{\nu h(u)} du, \tag{1.18}$$

$$H^{(2)}(c, \nu+1; \nu\lambda) = \frac{1}{2\pi i} \int_{\mathscr{L}_4} e^{\nu h(u)} du = \frac{1}{2\pi i} \int_{-\infty}^{\infty - i\pi} e^{\nu h(u)} du. \tag{1.19}$$

We consider the equation $h'(u) = 0$ and find the stationary point $u_0 = p_0 + iq_0$ as the root of following equation

$$\frac{d}{du} h(u) = \lambda \frac{1-c}{2} e^{\frac{1-c}{2}u} \sinh\left(\frac{1+c}{2}u\right) + \lambda \frac{1+c}{2} e^{\frac{1-c}{2}u} \cosh\left(\frac{1+c}{2}u\right) - 1 = 0, \tag{1.20}$$

which leads to

$$\lambda = \frac{x}{\nu} = \frac{2}{e^{u_0} + ce^{-cu_0}} = \frac{2}{(e^{p_0}\cos q_0 + ce^{-cp_0}\cos cq_0) + i\,(e^{p_0}\sin q_0 - ce^{-cp_0}\sin cq_0)}. \tag{1.21}$$

In comparison with (1.15), for $\lambda \in \mathbb{R}^+$ we establish the following the transcendental equation as the necessary condition for the stationary point $u_0 = p_0 + iq_0$

$$e^{p_0}\sin q_0 - ce^{-cp_0}\sin(cq_0) = 0, \quad c > -1, \tag{1.22}$$

and present the graph of the stationary points in Fig. 4 for the values of $-1 < c \leq 1$ and $c > 1$. We note that a conjugate property is valid between the stationary points. It means that, when $u_0 = p_0 + iq_0$ is a stationary point, then $u_0 = p_0 - iq_0$ is a stationary point too.

Remark 1.3. In the special case $c = 1$, the relation (1.22) is reduced to

$$\sinh(p_0)\sin(q_0) = 0, \tag{1.23}$$

which implies that for the case $p_0 = 0$, we have two stationary points $\pm iq_0$, and for the case $q_0 = 0$ we have the line $q = 0$ presented in Fig. 4.

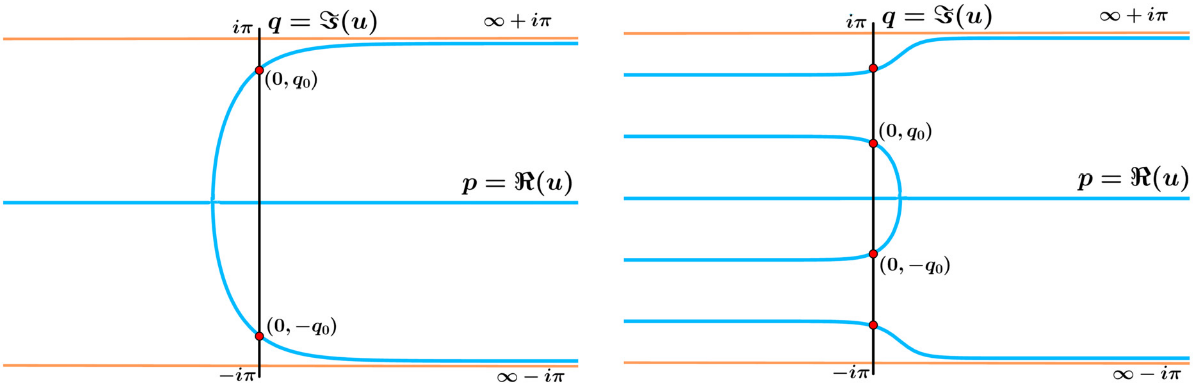


Fig. 4. The curves of stationary points for the equation (1.22) for $-1 < c \le 1$ (left) and $c > 1$ (right).

Now, in view of the necessary condition (1.22), we rewrite the parameter λ and the function h as follows

$$\lambda = \frac{2}{e^{p_0} \cos q_0 + c e^{-c p_0} \cos c q_0} \in \mathbb{R}^+, \tag{1.24}$$

$$h(u) = h(p,q) = \left[\frac{e^p \cos q - e^{-cp} \cos cq}{e^{p_0} \cos(q_0) + c e^{-cp_0} \cos(cq_0)} - p \right] + i \left[\frac{e^{-cp} \sin q + e^p \sin q}{e^{p_0} \cos(q_0) + c e^{-cp_0} \cos(cq_0)} - q \right]. \tag{1.25}$$

We apply the different strategies to find the steepest descent curve for finding the asymptotic expansion of Wright function in the stationary point $u_0 = p_0 + iq_0$. Our study is divided into two cases for $u_0 = p_0 + iq_0$ as the real point and the complex point. We organize the remaining of paper as follows. In Section 2, for the real stationary point $u_0 = p_0 > 0$ and $c > -1$, we find the asymptotic expansion of Wright function using the steepest descent and Lagrange theorems. In Section 3, the asymptotic expansions are concerned to the complex stationary point $u_0 = p_0 + iq_0$ and $c > 0$. All results generalize the asymptotic expansions of the Bessel functions of first and second kinds $J_\nu(z)$ and $Y_\nu(z)$

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu + k + 1)}, \tag{1.26}$$

$$Y_\nu(z) = \csc(\pi\nu)[J_\nu(z) \cos(\pi\nu) - J_{-\nu}(z)], \quad \nu \notin \mathbb{Z}. \tag{1.27}$$

Finally, the concluding remarks are given.

2. The real stationary point

In this case, for $c > -1$ we have the real stationary point $u_0 = p_0 \in \mathbb{R}$ and in this section, the relations (1.24) and (1.25) are simplified to

$$\lambda = \frac{2}{e^{p_0} + c e^{-cp_0}}, \tag{2.1}$$

$$h(u) = h(p,q) = \left[\frac{e^p \cos q - e^{-cp} \cos cq}{e^{p_0} + c e^{-cp_0}} - p \right] + i \left[\frac{e^{-cp} \sin q + e^p \sin q}{e^{p_0} + c e^{-cp_0}} - q \right]. \tag{2.2}$$

In this respect, we should generally mention that the steepest descent curves for $p_0 < 0$ does not follow the contour $\mathscr{L}$ and we ignore this case. Therefore, we state the following theorem for $p_0 \ge 0$.

Theorem 2.1. *For $c > -1$ and $p_0 \ge 0$, the following leading asymptotic term with an order estimate holds for the Wright function*

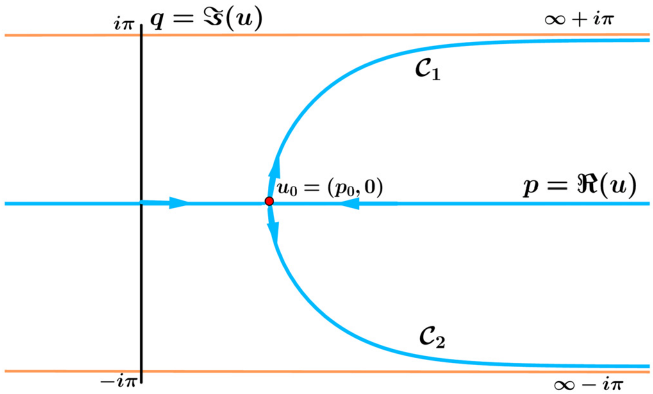


Fig. 5. The steepest descent curve $e^p \sin(q) + e^{-cp}\sin(cq) = q(e^{p_0} + ce^{-cp_0})$ for $c > -1$ and $p_0 \geq 0$.

$$\mathscr{W}(c, \nu+1; \lambda\nu) = \sqrt{\frac{e^{p_0} + ce^{-cp_0}}{2\pi\nu(e^{p_0} - c^2 e^{-cp_0})}}\, e^{\nu\left[\frac{e^{p_0} - e^{-cp_0}}{e^{p_0} + ce^{-cp_0}} - p_0\right]} \left(1 + O(\frac{1}{\sqrt{\nu}})\right), \quad \nu \to \infty, \tag{2.3}$$

where for $c \geq 1$, $p_0 \neq \frac{\ln(c^2)}{c+1}$.

Proof. In this case, the stationary point is $u_0 = p_0$ and consequently the relation (2.2) gives rise to

$$h(u_0) = \frac{e^{u_0} - e^{-cu_0}}{e^{p_0} + ce^{-cp_0}} - u_0. \tag{2.4}$$

In order to find the steepest descent curve, it is sufficient to consider the equation $\Im(h(u)) = \Im(h(u_0)) = 0$ and obtain the following relation for the desired curve [33]

$$e^p \sin(q) + e^{-cp}\sin(cq) = \left(e^{p_0} + ce^{-cp_0}\right) q, \quad c > -1, \quad p_0 \geq 0. \tag{2.5}$$

It is obvious that the curve (2.5) is symmetric with respect to the p-axis, when q varies from $-\pi$ to π, the conjugate contours $\mathscr{C}_1$ and $\mathscr{C}_2$ are begun from $u_0 = p_0$ and ended with $\infty \pm i\pi$, respectively. See Fig. 5 for the nonnegative part of p-axis. At this point, in view of Fig. 3 and (1.7), we can present

$$\mathscr{W}(c, \nu+1; \lambda\nu) = \frac{1}{2\pi i}\int_{\mathscr{C}_1} e^{\nu h(u)} du - \frac{1}{2\pi i}\int_{\mathscr{C}_2} e^{\nu h(u)} du, \quad \nu \to \infty, \tag{2.6}$$

with the following values for the function h

$$h(u) = \frac{e^u - e^{-cu}}{e^{p_0} + ce^{-cp_0}} - u, \quad h'(u) = \frac{e^u + ce^{-cu}}{e^{p_0} + ce^{-cp_0}} - 1, \quad h''(u) = \frac{e^u - c^2 e^{-cu}}{e^{p_0} + ce^{-cp_0}}. \tag{2.7}$$

Here, for $c > -1$ and $p_0 \geq 0$, we take into account the values

$$h'(p_0) = 0, \quad h''(p_0) \neq 0, \quad \alpha = \arg(h''(p_0)) = 0,$$

and establish the condition $p_0 \neq \frac{\ln(c^2)}{c+1}$ for $c \geq 1$. We also use the asymptotic expansion theorem to get

$$\int_{\mathscr{C}_1} e^{\nu h(u)} du = \sqrt{\frac{\pi(e^{p_0} + ce^{-cp_0})}{2\nu(e^{p_0} - c^2 e^{-cp_0})}}\, e^{\nu\left[\frac{e^{p_0} - e^{-cp_0}}{e^{p_0} + ce^{-cp_0}} - p_0\right] + i\theta_1} \left(1 + O(\frac{1}{\sqrt{\nu}})\right), \tag{2.8}$$

$$\int_{\mathscr{C}_2} e^{\nu h(u)} du = \sqrt{\frac{\pi(e^{p_0} + ce^{-cp_0})}{2\nu(e^{p_0} - c^2 e^{-cp_0})}} e^{\nu\left[\frac{e^{p_0} - e^{-cp_0}}{e^{p_0} + ce^{-cp_0}} - p_0\right] + i\theta_2} \left(1 + O(\frac{1}{\sqrt{\nu}})\right), \tag{2.9}$$

where θ_1 and θ_2 are the angels of stationary point $u_0 = p_0$ given by

$$\theta_1 = [-\frac{\alpha}{2} - \frac{\pi}{2}]_{\alpha=0} = -\frac{\pi}{2}, \quad \theta_2 = [-\frac{\alpha}{2} + \frac{\pi}{2}]_{\alpha=0} = \frac{\pi}{2}. \tag{2.10}$$

We finally substitute the above relations into (2.6) and complete the proof. □

In the next theorem, we apply the Lagrange expansion formula to establish an asymptotic expansion for the Wright function.

Theorem 2.2. *For $c > -1$ and $p_0 \geq 0$, the following asymptotic expansion holds for the Wright function*

$$\mathscr{W}(c, \nu+1; \lambda\nu) = \sqrt{\frac{e^{p_0} + ce^{-cp_0}}{2\pi\nu(e^{p_0} - c^2 e^{-cp_0})}} e^{\nu\left[\frac{e^{p_0} - e^{-cp_0}}{e^{p_0} + ce^{-cp_0}} - p_0\right]} \sum_{m=0}^{\infty} \frac{D_m}{\left(\nu(\frac{e^{p_0} - c^2 e^{-cp_0}}{e^{p_0} + ce^{-cp_0}})\right)^m}, \quad \nu \to \infty, \tag{2.11}$$

where for $c \geq 1$, $p_0 \neq \frac{\ln(c^2)}{c+1}$ and $D_m \in \mathbb{R}$ is given by

$$D_m = \frac{(-1)^m}{2^{2m+\frac{1}{2}} m!} \left(\frac{e^{p_0} - c^2 e^{-cp_0}}{e^{p_0} + ce^{-cp_0}}\right)^{m+\frac{1}{2}} \left[\frac{d^{2m}}{dw^{2m}} [\sum_{j=0}^{\infty} \mu_j w^j]^{-(m+\frac{1}{2})}\right]_{w=0}, \quad m = 0, 1, 2, \cdots, \tag{2.12}$$

such that

$$\begin{aligned} \mu_{2m} &= \frac{1}{(2m+2)!} \left[\frac{e^{p_0} - c^{2m+2} e^{-cp_0}}{e^{p_0} + ce^{-cp_0}}\right], \\ \mu_{2m+1} &= \frac{1}{(2m+3)!} \left[\frac{e^{p_0} + c^{2m+3} e^{-cp_0}}{e^{p_0} + ce^{-cp_0}}\right], \quad m = 0, 1, 2, \cdots. \end{aligned} \tag{2.13}$$

Proof. At first, we set the following map

$$\tau = h(p_0) - h(u) = \left[\frac{e^{p_0} - e^{-cp_0}}{e^{p_0} + ce^{-cp_0}} - p_0\right] - \left[\frac{e^{u} - e^{-cu}}{e^{p_0} + ce^{-cp_0}} - u\right], \tag{2.14}$$

and use the relation (2.6) to show

$$\mathscr{W}(c, \nu+1; \lambda\nu) = \frac{e^{\nu\left[\frac{e^{p_0} - e^{-cp_0}}{e^{p_0} + ce^{-cp_0}} - p_0\right]}}{2\pi i} \int_{\mathscr{C}_1{}' - \mathscr{C}_2{}'} e^{-\nu\tau} \frac{du}{d\tau} d\tau, \tag{2.15}$$

where $\mathscr{C}_1'$ and $\mathscr{C}_2'$ are the mapped contours in the τ-plane corresponding to the contours $\mathscr{C}_1$ and $\mathscr{C}_2$ in the u-plane, respectively. In this case, for the steepest curves $\mathscr{C}_1'$ and $\mathscr{C}_2'$ we have $\Im[\tau] = 0$. Therefore, we deduce that the contours $\mathscr{C}_1'$ and $\mathscr{C}_2'$ are the real axis in the τ-plane. Here, we apply the Taylor expansion and present

$$\tau = -\frac{1}{2!}\left[\frac{e^{p_0} - c^2 e^{-cp_0}}{e^{p_0} + ce^{-cp_0}}\right](u-p_0)^2 - \frac{1}{3!}\left[\frac{e^{p_0} + c^3 e^{-cp_0}}{e^{p_0} + ce^{-cp_0}}\right](u-p_0)^3$$
$$- \frac{1}{4!}\left[\frac{e^{p_0} - c^4 e^{-cp_0}}{e^{p_0} + ce^{-cp_0}}\right](u-p_0)^4 + \cdots, \tag{2.16}$$

or equivalently

$$\tau = -(u-p_0)^2\left[\mu_0 + \mu_1(u-p_0) + \mu_2(u-p_0)^2 + \cdots\right], \tag{2.17}$$

where the coefficient $\mu_j \in \mathbb{R}$ was shown in (2.13). Also, by setting $w = u - p_0$ we can obtain the following relation for employing the Lagrange expansion

$$u - p_0 = \pm i\tau^{\frac{1}{2}}\left[\mu_0 + \mu_1 w + \mu_2 w^2 + \cdots\right]^{-\frac{1}{2}}, \tag{2.18}$$

where the positive sign in the above relation corresponds to $\mathscr{C}_1$ $(q > 0)$ and the negative sign corresponds to $\mathscr{C}_2$ $(q < 0)$. In this stage, using Lagrange's theorem we arrive at

$$u - p_0 = \sum_{n=1}^{\infty}\frac{(\pm i)^n\tau^{\frac{n}{2}}}{n!}\left[\frac{d^{n-1}}{dw^{n-1}}[\mu_0 + \mu_1(u-p_0) + \mu_2(u-p_0)^2 + \cdots]^{-\frac{n}{2}}\right]_{w=0}. \tag{2.19}$$

We now substitute (2.19) into (2.15) and consider the inverse functions u_+ and u_- (related to the positive and negative signs, respectively) and apply the Watson lemma to obtain

$$\begin{aligned}
\int_{\mathscr{L}'} e^{-\nu\tau}\frac{du}{d\tau}d\tau &= \int_0^\infty e^{-\nu\tau}\frac{d(u_+ - u_-)}{d\tau}d\tau \\
&= i\int_0^\infty e^{-\nu\tau}\sum_{m=0}^{\infty}\frac{(-1)^m a_{2m+1}}{(2m)!}\tau^{m-\frac{1}{2}}d\tau \\
&\sim i\sum_{m=0}^{\infty}\frac{(-1)^m a_{2m+1}}{(2m)!}\Gamma(m+\frac{1}{2})\nu^{-(m+\frac{1}{2})} \\
&= i\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\frac{1}{2})a_{2m+1}}{2^m m!}\nu^{-(m+\frac{1}{2})},
\end{aligned} \tag{2.20}$$

where a_n is given by

$$a_n = \left[\frac{d^{n-1}}{dw^{n-1}}[\mu_0 + \mu_1 w + \mu_2 w^2 + \cdots]^{-\frac{n}{2}}\right]_{w=0}, \quad n = 1, 2, \cdots, \tag{2.21}$$

or briefly

$$a_1 = \mu_0^{-\frac{1}{2}} = \left(2\frac{e^{p_0} + ce^{-cp_0}}{e^{p_0} - c^2e^{-cp_0}}\right)^{\frac{1}{2}}, \tag{2.22}$$

$$a_2 = -\mu_0^{-2}\mu_1 = -\frac{2}{3}\left(\frac{e^{2p_0} + (c^3 + c)e^{(1-c)p_0} + c^4e^{-2cp_0}}{e^{2p_0} - 2c^2e^{(1-c)p_0} + c^4e^{-2cp_0}}\right), \tag{2.23}$$

$$\vdots \quad .$$

Finally, we obtain

$$\mathscr{W}(c,\nu+1;\lambda\nu)=\frac{e^{\nu\left[\dfrac{e^{p_0}-e^{-cp_0}}{e^{p_0}+ce^{-cp_0}}-p_0\right]}}{2\sqrt{\nu\pi}}\sum_{m=0}^{\infty}\frac{(-1)^m a_{2m+1}}{\nu^m 2^{2m} m!},\quad \nu\to\infty, \tag{2.24}$$

which by putting

$$D_m=\frac{(-1)^m}{2^{2m+\frac{1}{2}}m!}\left(\frac{e^{p_0}-c^2e^{-cp_0}}{e^{p_0}+ce^{-cp_0}}\right)^{m+\frac{1}{2}}a_{2m+1}, \tag{2.25}$$

we present (2.11) and complete the proof. □

Remark 2.3. In the special case $c=1$ and in view of (1.4), the relation (2.1) is equivalent to $\lambda=\frac{1}{\cosh(p_0)}<1$. For this case, the asymptotic expansion of the Bessel function J_ν is established as follows [34, p. 389]

$$J_\nu\Big(\frac{\nu}{\cosh(p_0)}\Big)=\frac{1}{\sqrt{2\pi\nu\tanh(p_0)}}e^{\nu\left(\tanh(p_0)-p_0\right)}\left(1+\frac{\frac{1}{8}-\frac{5}{24}\coth^2(p_0)}{\nu\tanh(p_0)}+\cdots\right),\quad \nu\to\infty. \tag{2.26}$$

Also, in the special case $c=0$, we can find $D_0=1$ and $D_1=-\frac{1}{12}$ and get the following asymptotic expansion for the exponential function

$$\mathscr{W}(0,\nu+1;x)=\frac{(\frac{1}{2}x)^\nu}{\Gamma(\nu+1)}e^{-\frac{x}{2}}=\frac{1}{\sqrt{2\pi\nu}}\frac{(\frac{1}{2}x)^\nu}{\nu^\nu}e^{\nu-\frac{x}{2}}(1-\frac{1}{12\nu}+\cdots),\quad \nu\to\infty. \tag{2.27}$$

Remark 2.4. In order to study the numerical verification, we consider the leading term of (2.11) and show the graphs of the relative errors for different values of c and $\nu=100$ in Fig. 6. It is obvious that these presentations refer to the suitable approximations with the very small errors. We should also mention that the relative error increases near the point $p_0=\frac{\ln(c^2)}{c+1}$.

3. The complex stationary point

In this case, for $c>0$ we have the conjugate stationary points $p_0\pm iq_0$ which has been demonstrated in Fig. 4. In this sense, we have

$$\lambda=\frac{2}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)},\quad q_0\neq 0, \tag{3.1}$$

and mention that the used approach is not valid for the case $-1<c<0$. Therefore, we only restrict ourselves to the case $c>0$ and recall the relations (1.24) and (1.25) for stating the following theorem.

Theorem 3.1. *For $c>0$ and $u_0=p_0+iq_0$, the following leading asymptotic term with an order estimate holds for the Wright function*

$$\mathscr{W}(c,\nu+1;\lambda\nu)=\Re\left[\frac{2e^{\nu h(u_0)-i\frac{\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\left(1+O(\frac{1}{\sqrt{\nu}})\right)\right],\quad \alpha=\arg(h''(u_0)), \tag{3.2}$$

where

$$u_0\neq\frac{\ln(c^2)}{1+c}+i\frac{2k\pi}{1+c},\quad c+1>2k,\quad k\in\mathbb{N}. \tag{3.3}$$

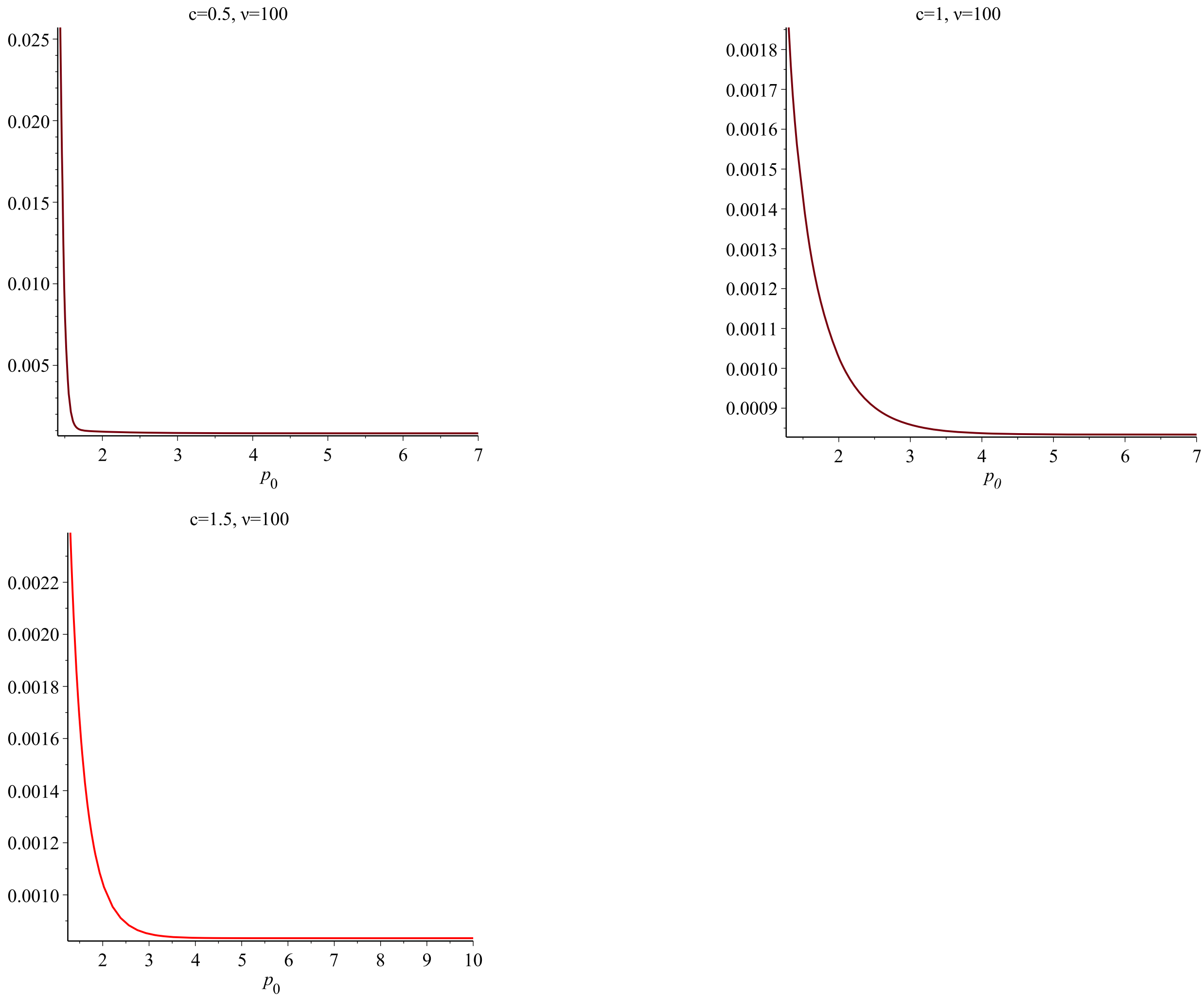


Fig. 6. The graphs of the relative errors for the relation (2.11).

Proof. In order to find the steepest descent curve, it is sufficient to set $\Im(h(u)) = \Im(h(u_0))$ and obtain following equation

$$e^p \sin q + e^{-cp} \sin(cq) = \left[e^{p_0} \cos(q_0) + ce^{-cp_0} \cos(cq_0)\right](q - q_0) + \left[\sin(q_0) + \sin(cq_0)\right]. \tag{3.4}$$

The curve (3.4) for the conjugate stationary points $p_0 \pm iq_0$ has been plotted in Fig. 7. The steepest descent curve for $u_0 = p_0 \pm iq_0$ is symmetric with respect to the p-axis. For this case, the curve $\mathscr{C}_5$ is conjugate of $\mathscr{C}_3$, and the curve $\mathscr{C}_6$ is conjugate of $\mathscr{C}_4$. Moreover, the curve $\mathscr{C}_3$ is begun from $u_0 = p_0 + iq_0$ and ended with $\infty + i\pi$, and the curve $\mathscr{C}_4$ is begun from $u_0 = p_0 + iq_0$ and ended with $-\infty$. Now, in view of the relation (1.8) and Fig. 3, we have

$$H^{(1)}(c, \nu + 1; \lambda\nu) = \frac{1}{\pi i} \int\limits_{\mathscr{C}_3 - \mathscr{C}_4} e^{\nu h(u)} du, \quad \nu \to \infty, \tag{3.5}$$

$$H^{(2)}(c, \nu + 1; \lambda\nu) = \frac{1}{\pi i} \int\limits_{\mathscr{C}_6 - \mathscr{C}_5} e^{\nu h(u)} du, \quad \nu \to \infty, \tag{3.6}$$

with the following values for the function h

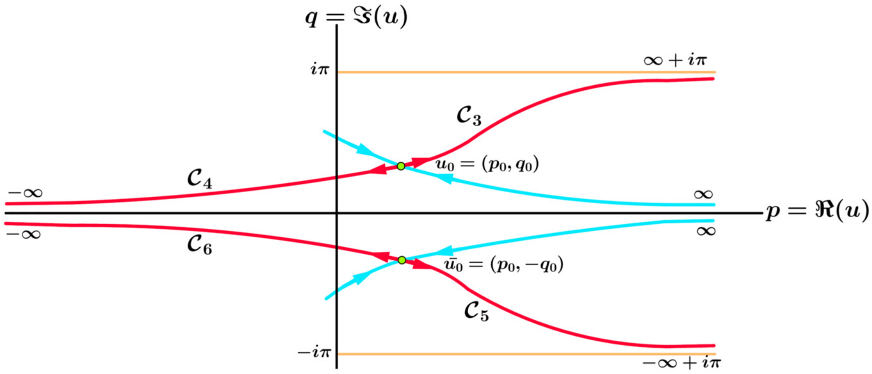


Fig. 7. The steepest descent curves for the relation (3.4).

$$h(u) = \frac{e^u - e^{-cu}}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} - u, \tag{3.7}$$

$$h'(u) = \frac{e^u + ce^{-cu}}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} - 1, \tag{3.8}$$

$$h''(u) = \frac{e^u - c^2e^{-cu}}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)}, \tag{3.9}$$

such that $h'(\overline{u_0}) = \overline{h'(u_0)}$ and $h'(u_0) = h'(p_0 + iq_0) = 0$. Also, for the second derivative of the function h, we have

$$h''(u_0) = \left(\frac{e^{p_0}\cos(q_0) - c^2e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)}\right) + i\left(\frac{e^{p_0}\sin(q_0) + c^2e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)}\right), \tag{3.10}$$

such that $h''(\overline{u_0}) = \overline{h''(u_0)}$ and $h''(u_0) = h''(p_0 + iq_0) \neq 0$ (it is clear that the value of $h''(u_0)$ vanishes at the point $u_0 = \frac{\ln(c^2)}{1+c} + i\frac{2k\pi}{1+c}, k \in \mathbb{N}$). Now, by considering the argument

$$\alpha = \arg(h''(u_0)) = \arctan\left(\frac{e^{p_0}\sin(q_0) + c^2e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) - c^2e^{-cp_0}\cos(cq_0)}\right), \tag{3.11}$$

we can establish the following asymptotic behaviors

$$\int\limits_{\mathscr{C}_j} e^{\nu h(u)}du = \sqrt{\frac{\pi}{2\nu|h''(u_0)|}}e^{\nu h(u_0)+i\theta_j}\left(1 + O(\frac{1}{\sqrt{\nu}})\right), \quad j = 3, 4, \tag{3.12}$$

$$\int\limits_{\mathscr{C}_j} e^{\nu h(u)}du = \sqrt{\frac{\pi}{2\nu|h''(\bar{u_0})|}}e^{\nu h(\bar{u_0})+i\theta_j}\left(1 + O(\frac{1}{\sqrt{\nu}})\right), \quad j = 5, 6, \tag{3.13}$$

where the angles θ_j are related to the contours $\mathscr{C}_j$ as

$$\theta_3 = \frac{\pi - \alpha}{2}, \quad \theta_4 = \frac{3\pi - \alpha}{2}, \quad \theta_5 = \frac{\alpha - \pi}{2}, \quad \theta_6 = \frac{\alpha - 3\pi}{2}. \tag{3.14}$$

We now substitute (3.12), (3.13) into (3.5), (3.6), respectively, and get

$$H^{(1)}(c, \nu + 1; \lambda\nu) = \frac{e^{\nu h(u_0)}\left(e^{i\theta_3} - e^{i\theta_4}\right)}{i\sqrt{2\pi\nu|h''(u_0)|}}\left(1 + O(\frac{1}{\sqrt{\nu}})\right), \tag{3.15}$$

$$H^{(2)}\left(c,\nu+1;\lambda\nu\right)=\frac{e^{\nu h(\bar{u_0})}\left(e^{i\theta_6}-e^{i\theta_5}\right)}{i\sqrt{2\pi\nu|h''(\bar{u_0})|}}\left(1+O(\frac{1}{\sqrt{\nu}})\right). \tag{3.16}$$

At this point, using the following facts

$$|h''(\bar{u})|=|\overline{h''(u)}|=|h''(u)|, \tag{3.17}$$

and the relationships between the angles $\theta_5=-\theta_3,\theta_6=-\theta_4$, we arrive at

$$H^{(2)}\left(c,\nu+1;\lambda\nu\right)=\frac{e^{\nu\overline{h(u_0)}}\left(e^{-i\theta_3}-e^{-i\theta_4}\right)}{-i\sqrt{2\pi\nu|h''(u_0)|}}\left(1+\overline{O(\frac{1}{\sqrt{\nu}})}\right). \tag{3.18}$$

Thus, we can deduce

$$H^{(2)}\left(c,\nu+1;\lambda\nu\right)=\overline{H^{(1)}\left(c,\nu+1;\lambda\nu\right)}, \tag{3.19}$$

where

$$H^{(1)}\left(c,\nu+1;\lambda\nu\right)=\frac{e^{\nu h(u_0)}\left(e^{i(\theta_3-\frac{\pi}{2})}-e^{i(\theta_4-\frac{\pi}{2})}\right)}{\sqrt{2\pi\nu|h''(u_0)|}}\left(1+O(\frac{1}{\sqrt{\nu}})\right) \tag{3.20}$$

$$=\frac{2e^{\nu h(u_0)-i\frac{\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\left(1+O(\frac{1}{\sqrt{\nu}})\right). \tag{3.21}$$

Finally, by taking into account the relation (1.6), we obtain (3.2) from the following relation

$$\mathscr{W}\left(c,\nu+1;\lambda\nu\right)=\Re\left(H^{(1)}\left(c,\nu+1;\lambda\nu\right)\right), \tag{3.22}$$

and complete the proof. □

Corollary 3.2. *By the same procedure to Theorem 3.1, for generalization of the Bessel function of second kind Y_ν presented by*

$$\mathscr{Y}(c,d;z)=\frac{1}{2i}\left[H^{(1)}(c,d;z)-H^{(2)}(c,d;z)\right], \tag{3.23}$$

we can state the following leading asymptotic term with an order estimate

$$\mathscr{Y}\left(c,\nu+1;\lambda\nu\right)=\Im\left[\frac{2e^{\nu h(u_0)-i\frac{\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\left(1+O(\frac{1}{\sqrt{\nu}})\right)\right]. \tag{3.24}$$

Theorem 3.3. *For $c>0$, the following asymptotic expansion holds for the Wright function*

$$\mathscr{W}\left(c,\nu+1;\lambda\nu\right)=\Re\left[\frac{2e^{\nu h(u_0)-\frac{i\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\sum_{m=0}^{\infty}\frac{D'_m}{\left(\nu h''(u_0)\right)^m}\right],\quad \alpha=\arg(h''(u_0)), \tag{3.25}$$

where

$$u_0\neq\frac{\ln(c^2)}{1+c}+i\frac{2k\pi}{1+c},\quad c+1>2k,\quad k\in\mathbb{N}, \tag{3.26}$$

and

$$D'_m = \frac{(-1)^m (h''(u_0))^{m+\frac{1}{2}}}{2^{2m+\frac{1}{2}}} \left[\frac{d^{2m}}{dw^{2m}} [\sum_{j=0}^{\infty} \eta_j w^j]^{-(m+\frac{1}{2})} \right]_{w=0}, \ m = 0, 1, 2, \cdots, \tag{3.27}$$

such that

$$\eta_{2m} = \frac{1}{(2m+2)!} \left[\left(\frac{e^{p_0}\cos(q_0) - c^{2m+2}e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) + i \left(\frac{e^{p_0}\sin(q_0) + c^{2m+2}e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) \right], \tag{3.28}$$

$$\eta_{2m+1} = \frac{1}{(2m+3)!} \left[\left(\frac{e^{p_0}\cos(q_0) + c^{2m+3}e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) + i \left(\frac{e^{p_0}\sin(q_0) - c^{2m+3}e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) \right]. \tag{3.29}$$

Proof. At first, we consider

$$h(u_0) = \left[\frac{e^{p_0}\cos(q_0) - e^{-cp_0}\cos(cq_0)}{e^{p_0\cos(q_0)} + ce^{-cp_0}\cos(cq_0)} - p_0 \right] + i \left[\frac{e^{p_0}\sin(q_0) + e^{-cp}\sin(cq_0)}{e^{p_0\cos(q_0)} + ce^{-cp_0}\cos(cq_0)} - q_0 \right], \tag{3.30}$$

and set the map $\tau = h(u_0) - h(u)$. In this sense, we use the relations (1.25), (3.5) and (3.6) to show

$$H^{(1)}(c, \nu+1; \lambda\nu) = \frac{1}{\pi i} e^{\nu h(u_0)} \int_{\mathscr{C}'_3 - \mathscr{C}'_4} e^{-\nu\tau} \frac{du}{d\tau} d\tau, \tag{3.31}$$

$$H^{(2)}(c, \nu+1; \lambda\nu) = \frac{1}{\pi i} e^{\nu \overline{h(u_0)}} \int_{\mathscr{C}'_6 - \mathscr{C}'_5} e^{-\nu\tau} \frac{du}{d\tau} d\tau, \tag{3.32}$$

where $\mathscr{C}'_j$, $j = 3, 4, 5, 6$, are the contours in the τ-plane corresponding to $\mathscr{C}_j$ in the u-plane. In this case, $\Im(h(u))$ is nonzero and equal to the constant value $\Im(h(u_0))$ for $j = 3, 4$, and equal to $-\Im(h(u_0))$ for $j = 5, 6$. In this regard, by using the Taylor expansion we have

$$\begin{aligned}
\tau = h(u_0) - h(u) &= -\frac{1}{2!} h''(u_0)(u-u_0)^2 - \frac{1}{3!} h^{(3)}(u_0)(u-u_0)^3 - \frac{1}{4!} h^{(4)}(u_0)(u-u_0)^4 - \cdots \\
&= -\frac{1}{2!} \left[\left(\frac{e^{p_0}\cos(q_0) - c^2 e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) + i \left(\frac{e^{p_0}\sin(q_0) + c^2 e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) \right] (u - (p_0 + iq_0))^2 \\
&- \frac{1}{3!} \left[\left(\frac{e^{p_0}\cos(q_0) + c^3 e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) + i \left(\frac{e^{p_0}\sin(q_0) - c^3 e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) \right] (u - (p_0 + iq_0))^3 \\
&- \frac{1}{4!} \left[\left(\frac{e^{p_0}\cos(q_0) - c^4 e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) + i \left(\frac{e^{p_0}\sin(q_0) + c^4 e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0) + ce^{-cp_0}\cos(cq_0)} \right) \right] (u - (p_0 + iq_0))^4 \\
&- \ldots,
\end{aligned} \tag{3.33}$$

or equivalently

$$\tau = -(u - (p_0 + iq_0))^2 \left[\eta_0 + \eta_1 (u - (p_0 + iq_0)) + \eta_2 (u - (p_0 + iq_0))^2 + \cdots \right], \tag{3.34}$$

where $\eta_j \in \mathbb{C}$ are obtained via (3.28) and (3.29). Now, in view of the contours $\mathscr{C}_3$ and $\mathscr{C}_4$ which correspond to the arguments $\arg(\eta_0) = \theta_3$ and $\arg(\eta_0) = \pi + \theta_3 = \theta_4$, respectively, we show the following representation

$$u - (p_0 + iq_0) = \pm i \tau^{\frac{1}{2}} \left[\eta_0 + \eta_1 w + \eta_2 w^2 + \cdots \right]^{-\frac{1}{2}}, \tag{3.35}$$

and employ the Lagrange expansion formula to find

$$u-(p_0+iq_0)=\sum_{n=1}^{\infty}\frac{(\pm i)^n\tau^{\frac{n}{2}}}{n!}\left[\frac{d^{n-1}}{dw^{n-1}}[\eta_0+\eta_1 w+\mu_2 w^2+\cdots]^{-\frac{n}{2}}\right]_{w=0}. \tag{3.36}$$

We substitute (3.36) into (3.31) and consider the inverse functions u_+ and u_- to apply the Watson lemma as

$$\begin{aligned}\int\limits_{\mathscr{C}_3'-\mathscr{C}_4'} e^{-\nu\tau}\frac{du}{d\tau}d\tau &= \int\limits_0^\infty e^{-\nu\tau}\frac{d(u_+-u_-)}{d\tau}d\tau\\ &= i\int\limits_0^\infty e^{-\nu\tau}\sum_{m=0}^{\infty}\frac{(-1)^m b_{2m+1}}{(2m)!}\tau^{m-\frac{1}{2}}d\tau\\ &\sim i\sum_{m=0}^{\infty}\frac{(-1)^m b_{2m+1}}{(2m)!}\Gamma(m+\frac{1}{2})\nu^{-(m+\frac{1}{2})}\\ &= i\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\frac{1}{2})b_{2m+1}}{2^{2m}m!}\nu^{-(m+\frac{1}{2})},\end{aligned} \tag{3.37}$$

where the coefficient b_n is given by

$$b_n=\left[\frac{d^{n-1}}{dw^{n-1}}[\eta_0+\eta_1 w+\eta_2 w^2+\cdots]^{-\frac{n}{2}}\right]_{w=0},\quad n=1,2,\cdots, \tag{3.38}$$

or briefly

$$b_1=\eta_0^{-\frac{1}{2}}=\sqrt{2}\left[\frac{e^{p_0}\cos(q_0)-c^2e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}+i\frac{e^{p_0}\sin(q_0)+c^2e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}\right]^{-\frac{1}{2}}, \tag{3.39}$$

$$\begin{aligned}b_2=-\eta_0^{-2}\eta_1=-\frac{2}{3}&\left[\frac{e^{p_0}\cos(q_0)-c^2e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}+i\frac{e^{p_0}\sin(q_0)+c^2e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}\right]^{-2}\\ \times&\left[\frac{e^{p_0}\cos(q_0)+c^3e^{-cp_0}\cos(cq_0)}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}+i\frac{e^{p_0}\sin(q_0)-c^3e^{-cp_0}\sin(cq_0)}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}\right],\end{aligned} \tag{3.40}$$

$$\vdots \quad .$$

Consequently, we obtain

$$H^{(1)}(c,\nu+1;\lambda\nu)=\frac{e^{\nu h(u_0)}}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\frac{1}{2})b_{2m+1}}{2^{2m}\nu^{m+\frac{1}{2}}m!}, \tag{3.41}$$

or equivalently

$$H^{(1)}(c,\nu+1;\lambda\nu)=\frac{2e^{\nu h(u_0)-\frac{i\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\sum_{m=0}^{\infty}\frac{D_m'}{(\nu h''(u_0))^m}, \tag{3.42}$$

where

$$D_m'=\frac{(-1)^m(h''(u_0))^{m+\frac{1}{2}}b_{2m+1}}{2^{2m+\frac{1}{2}}},\quad h''(u_0)=|h''(u_0)|e^{i\alpha}. \tag{3.43}$$

Similarly, in view of the conjugate stationary point $\overline{u_0}=p_0-iq_0$, we can present

$$H^{(2)}\left(c,\nu+1;\lambda\nu\right)=\overline{H^{(1)}\left(c,\nu+1;\lambda\nu\right)}=\frac{2e^{\nu\overline{h(u_0)}+\frac{i\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\sum_{m=0}^{\infty}\frac{\overline{D'_m}}{\left(\nu\overline{h''(u_0)}\right)^m}. \tag{3.44}$$

Finally, we use (1.6) to get

$$\mathscr{W}\left(c,\nu+1;\lambda\nu\right)=\Re\left[\frac{2e^{\nu h(u_0)-\frac{i\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\sum_{m=0}^{\infty}\frac{D'_m}{\left(\nu h''(u_0)\right)^m}\right], \tag{3.45}$$

and complete the proof. □

Corollary 3.4. *By the same procedure to Theorem 3.3, for generalization of the Bessel function of second kind Y_ν we can state the following asymptotic expansion*

$$\mathscr{Y}\left(c,\nu+1;\lambda\nu\right)=\Im\left[\frac{2e^{\nu h(u_0)-\frac{i\alpha}{2}}}{\sqrt{2\pi\nu|h''(u_0)|}}\sum_{m=0}^{\infty}\frac{D'_m}{\left(\nu h''(u_0)\right)^m}\right], \tag{3.46}$$

where D'_m and η_m are given by (3.27), (3.28) and (3.29).

Remark 3.5. In the special case $c=1$ and $u_0=iq_0$, we have $h(u_0)=i\left(\tan(q_0)-q_0\right)$, $h''(u_0)=i\tan(q_0)$ and $\lambda=\frac{1}{\cos(q_0)}>1$. Thus, an alternative asymptotic expansion of the Bessel function J_ν is also established as follows [34, p. 392]

$$\begin{aligned}J_\nu(\nu\sec(q_0))&=\frac{2}{\sqrt{2\pi\nu\tan(q_0)}}\cos\left(\nu\tan(q_0)-\nu q_0-\frac{\pi}{4}\right)\\&+\frac{\frac{1}{8}+\frac{5}{24}\cot^2(q_0)}{\nu\tan(q_0)}\frac{2}{\sqrt{2\pi\nu\tan(q_0)}}\sin\left(\nu\tan(q_0)-\nu q_0-\frac{\pi}{4}\right)+\cdots,\quad \nu\to\infty.\end{aligned} \tag{3.47}$$

Remark 3.6. In order to discuss the numerical verification of (3.25), we should choose the points (p_0,q_0) such that the condition (1.22) along with the relation (3.1) hold (for $\lambda>0$). In order to get a suitable approximation, we can not follow our computations near the points $u_0=\frac{\ln(c^2)}{1+c}+i\frac{2k\pi}{1+c}$, $c+1>2k$, $k\in\mathbb{N}$. Summarizing, the point (p_0,q_0) should satisfy the following system

$$\begin{cases} e^{p_0}\sin q_0-ce^{-cp_0}\sin(cq_0)=0, & c>0,\\ \dfrac{2}{e^{p_0}\cos(q_0)+ce^{-cp_0}\cos(cq_0)}>0, & c>0,\\ p_0\neq\frac{\ln(c^2)}{1+c},\quad q_0\neq\frac{2k\pi}{1+c},\quad c+1>2k, & k\in\mathbb{N}.\end{cases} \tag{3.48}$$

4. Concluding remarks

In this paper, we considered a generalization of the Schläfli integral for the modification of Wright function. This integral was established by the exponential map on a contour consists of three sides of a rectangle. This contour played a fundamental role in determining the steepest descent curves for asymptotic expansion of the integral. Our discussion was summarized by two cases for the stationary points and associated steepest descent curves. In the all cases, we tried to find the steepest descent curves with respect to the Schläfli integral contour. In this sense, in Section 3 for the complex stationary point, we had to introduce new functions $H^{(j)}$, $j=1,2$, to present the desired curves tending to the Schläfli integral contour. Finally, the associated asymptotic expansions were given using the steepest descent and the Lagrange theorem.

Here, we should mention a critical point for the all obtained results. In the all computations, we considered a simple stationary point u_0 ($h''(u_0) \neq 0$) for presenting the asymptotic expansions. In the non-simple stationary point ($h''(u_0) = 0$) or near this point, we state the following remarks. In view of the necessary condition (1.22) for the steepest descent curve

$$e^{p_0} \sin(q_0) - ce^{-cp_0} \sin(cq_0) = 0, \tag{4.1}$$

and taking into account the condition

$$h''(u_0) = \frac{e^{u_0} - c^2 e^{-cu_0}}{e^{p_0} \cos(q_0) + ce^{-p_0} \cos(cq_0)} = 0, \quad u_0 = p_0 + iq_0, \tag{4.2}$$

we should solve these algebraic equations, simultaneously, and get the non-simple stationary points as

$$u_0 = \frac{\ln(c^2)}{1+c} + i\frac{2k\pi}{1+c}, \quad c+1 > 2k, \quad k = 0, 1, 2, \cdots . \tag{4.3}$$

When the stationary point is the exact point u_0, we should follow the discussion of Section 1 and apply Theorem 1.1 for $n = 3$ to represent the associated asymptotic expansions. See [33, §23.5] for the case $c = 1$ and representing the Airy-type expansion for the Bessel function $J_\nu(z)$. When the stationary point is near point $u_0 = \frac{\ln(c^2)}{1+c} + i\frac{2k\pi}{1+c}$, the parameter λ tends to

$$\lambda \to \frac{2}{c^{\frac{2}{1+c}} \cos(\frac{2k\pi}{1+c}) + c^{-\frac{2c}{1+c}} \cos(\frac{2kc\pi}{1+c})}, \quad c+1 > 2k, \quad k = 0, 1, 2, \cdots , \tag{4.4}$$

and in the vicinity of u_0, for $c \geq 1$ we have

$$\nu = z\left(\frac{1}{\lambda} - \varepsilon\right), \quad c \geq 1, \quad \varepsilon > 0, \quad \nu \in \mathbb{C}, \quad z = xe^{i\varphi}, \quad |\varphi| = |\arg(z)| < \frac{\pi}{2}. \tag{4.5}$$

In this respect, we can consider the following integral

$$\begin{aligned} \mathscr{W}(c, \nu+1; z) &= \frac{1}{2\pi i} \int_{\mathscr{L}} e^{xe^{i\varphi}\left[e^{\frac{1-c}{2}u} \sinh(\frac{1+c}{2}u) - \frac{u}{\lambda}\right] + z\varepsilon u} du, \\ &= \frac{1}{2\pi i} \int_{\infty - i\pi}^{\infty + i\pi} e^{xe^{i\varphi}\left[e^{\frac{1-c}{2}u} \sinh(\frac{1+c}{2}u) - \frac{u}{\lambda}\right] + z\varepsilon u} du, \\ & c \geq 1, \quad \varepsilon > 0, \quad \nu \in \mathbb{C}, \quad x \in \mathbb{R}, \quad |\varphi| < \frac{\pi}{2}, \end{aligned} \tag{4.6}$$

and take into account the functions $h(u) = e^{i\varphi}\left[e^{\frac{1-c}{2}u} \sinh(\frac{1+c}{2}u) - \frac{u}{\lambda}\right]$ and $g(u) = e^{z\varepsilon u}$. Based on the steepest descent method, if we now follow the proposed approach in [34, §7.12], then we can approximate the integral (4.6) with an upper bound near u_0 presented in Figs. 5 and 7. The point $u_0 = p_0 + iq_0$ in these figures are mixed in Fig. 8 for the case $c = 1$ and $u_0 = 0$ [34, p. 395]. We omit the associated details in this case.

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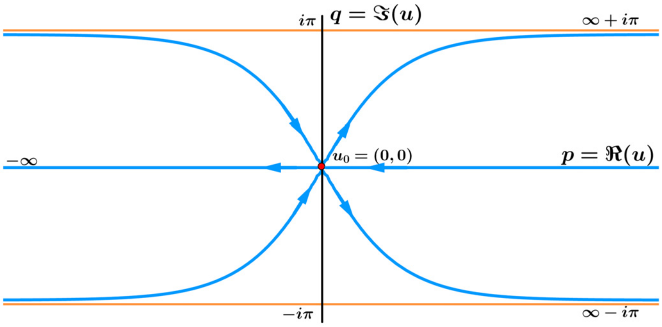

Fig. 8. The steepest decent curve $\cosh(p)\sin(q) = q$ for $c = 1$.

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