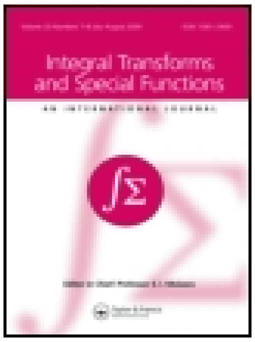

Integral Transforms and Special Functions

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/gitr20

Lebedev identities and integral representations of products of Hermite functions

Alireza Ansari & Hassan Askari

INTEGRAL TRANSFORMS AND SPECIAL FUNCTIONS
https://doi.org/10.1080/10652469.2021.1924704

Taylor & Francis
Taylor & Francis Group

Research Article

Check for updates

Lebedev identities and integral representations of products of Hermite functions

Alireza Ansari and Hassan Askari

Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, Shahrekord, Iran

ABSTRACT
In this paper, we consider the third-order differential equation of the quadratic products of Hermite functions and present the integral representations of products of Hermite functions (Lebedev identities) in terms of the confluent hypergeometric functions. Manipulating the integrands of these identities with various integral representations lead us to get new representations for the products of Hermite functions.

ARTICLE HISTORY
Received 14 February 2021
Accepted 28 April 2021

KEYWORDS
Hermite function; confluent hypergeometric function; Laplace transform

MATHEMATICS SUBJECT CLASSIFICATIONS 2020
33C15; 33C45; 44A10

1. Introduction

The Hermite functions $H_\nu(z)$ and $H_\nu(-z)$ of degree ν have a pivotal role in quantum mechanics [1] and are considered as the two linear independent solutions of the well-known differential equation [2, §10.2]

$$y'' - 2zy' + 2\nu y = 0, \quad \nu \in \mathbb{R},\ \nu \neq 0, 1, 2, \ldots . \tag{1.1}$$

Also, the Hermite functions can be expressed in terms of the parabolic cylinder functions

$$D_\nu(z) = 2^{-\frac{\nu}{2}} \mathrm{e}^{-\frac{z^2}{4}} H_\nu\left(\frac{z}{\sqrt{2}}\right), \tag{1.2}$$

and can be reduced to the Hermite polynomials for the case $\nu = n = 0, 1, 2, \ldots$. The establishing of integral representations for the products of parabolic cylinder functions have been taken into considerations by some authors in the literature [3–9]. These representations for the products of the parabolic cylinder functions are the existing developments and were mostly constructed as the Laplace transforms of some special functions such as the confluent hypergeometric functions and the parabolic cylinder functions. As an interesting remark in these representations, we can point out that the indices and arguments of the products of parabolic cylinder functions were shown as the parameters of associated

CONTACT Alireza Ansari ansari.alireza@sku.ac.ir, alireza_1038@yahoo.com Department of Applied Mathematics, Faculty of Mathematical Sciences, Shahrekord University, P.O. Box 115, Shahrekord, Iran

Laplace transforms. In this paper, in view of the integral representation of $H_\nu(z)$ [2]

$$H_\nu(z) = \frac{1}{\Gamma(-\nu)} \int_0^\infty \mathrm{e}^{-t^2-2tz} \frac{\mathrm{d}t}{t^{\nu+1}}, \quad \Re(\nu) < 0, \ |\arg(z)| < \pi, \tag{1.3}$$

we intend to find new integral representations for the quadratic products of Hermite functions in terms of the confluent hypergeometric functions. For this purpose, we begin with a third-order differential equation and apply the Laplace-type integral technique to present the function $H_\nu^2(z)$ by a Laplace integral representation. Using (1.3), we also establish the associated double integrals and make the suitable transformations to get other representations for the products of Hermite functions. We focus on the differential equation (this ODE is derived from [10, p. 512, 3.26])

$$y''' - 6zy'' + (8z^2 + 8\nu - 2)y' - 16\nu zy = 0, \quad \Re(\nu) < 0, \ |\arg(z)| < \pi, \tag{1.4}$$

with the general solution

$$y = c_1 H_\nu^2(z) + c_2 H_\nu(z) H_\nu(-z) + c_3 H_\nu^2(-z), \tag{1.5}$$

and seek for the integral representations for the three linear independent solutions in Section 2. In [2], for the first time, Lebedev showed a representation for $H_\nu^2(z)$ and we follow this way to obtain other identities similar to it. We get three integral representations for the products of the Hermite functions and we call them as the 'Lebedev identities'. Later, Nasri represented these identities via the operational calculus of Laplace transforms for the products of parabolic cylinder functions [9]. In Section 3, we use a simple technique [11–13] and manipulate the integrands of theses identities with various integral representations and derive new representations for the products of Hermite functions. The Laplace transform is the complementary tool to give an indicative direct for our approach and establishing integral representations in terms of the parabolic cylinder and the hypergeometric functions.

2. The Lebedev identities

2.1. The Laplace integral

In this section, we consider the Laplace-type integral and substitute the function y into the differential equation (1.4)

$$y(z) = \int_{\mathfrak{L}} \mathrm{e}^{-tz} s(t)\, \mathrm{d}t, \tag{2.1}$$

where $\mathfrak{L}$ is the Laplace contour which should be determined. We get the following relations for obtaining the function $s(t)$ and the contour $\mathfrak{L}$

$$\begin{cases} 8ts''(t) + \left(6t^2 + 16\nu + 16\right)s'(t) + \left(t^3 + (10 + 8\nu)t\right)s(t) = 0, \\ \left[\left(4ts'(t) + \left(4zt + 3t^2 + 4 + 8\nu\right)s(t)\right) \mathrm{e}^{-tz}\right]_{\mathfrak{L}} = 0. \end{cases} \tag{2.2}$$

We solve the second-order differential equation in (2.2) and get the general solution in terms of the confluent hypergeometric functions of first and second kinds

$$s(t) = c_1\, \mathrm{e}^{-\frac{1}{4}t^2}\, \Phi\left(\frac{1}{2}, \frac{3}{2} + \nu; \frac{1}{8}t^2\right) + c_2\, \mathrm{e}^{-\frac{1}{4}t^2}\, \Psi\left(\frac{1}{2}, \frac{3}{2} + \nu; \frac{1}{8}t^2\right), \tag{2.3}$$

where the functions Φ and Ψ are presented by Lebedev [2, Chapter 9]

$$\Phi(\alpha,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 e^{z\xi}\xi^{\alpha-1}(1-\xi)^{\gamma-\alpha-1}\,d\xi, \quad \Re(\gamma) > \Re(\alpha) > 0, \tag{2.4}$$

$$= \sum_{k=0}^{\infty}\frac{(\alpha)_k}{(\gamma)_k}\frac{z^k}{k!}, \quad |z| < \infty,\ \gamma \neq 0,-1,-2,\ldots, \tag{2.5}$$

$$\Psi(\alpha,\gamma;z) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-z\xi}\xi^{\alpha-1}(1+\xi)^{\gamma-\alpha-1}\,d\xi, \quad \Re(\alpha) > 0,\ \Re(z) > 0, \tag{2.6}$$

$$= \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\Phi(\alpha,\gamma;z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}z^{1-\gamma}\Phi(1+\alpha-\gamma,2-\gamma;z),$$
$$|z| < \infty,\ \gamma \neq 0,\pm 1,\pm 2,\ldots, \tag{2.7}$$

and $(.)_k$ is the Pochhammer symbol. We now consider the representation of function Ψ in (2.6) to represent the desired solution (2.3) in the following form

$$y(z) = c_1'\int_{\mathfrak{L}} e^{-t^2-2zt}\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}t^2\right)dt$$
$$+ c_2'\int_{\mathfrak{L}} e^{-t^2-2zt}t^{-2\nu-1}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t^2\right)dt. \tag{2.8}$$

We choose the contour $\mathfrak{L}$ such that the second identity in (2.2) holds. Setting $t = \rho\, e^{i\theta}$, we see that the identity (2.8) is identically zero for any contour $\mathfrak{L}$ inside the region $|\theta| < \frac{\pi}{4}$. In this sense, we consider $\mathfrak{L} = (0,\infty)$ and get the integral representation for the quadratic product of Hermite functions as

$$H_\nu^2(z) = c_2'\int_0^{\infty} e^{-t^2-2zt}t^{-2\nu-1}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t^2\right)dt. \tag{2.9}$$

In order to determine the coefficient c_2', we can replace the confluent hypergeometric function in (2.9) by the leading term of its asymptotic expansion. Then, we apply the saddle-point method to get a new asymptotic expansion and compare with the asymptotic expansion of $H_\nu^2(z)$. Finally, we get the coefficient $c_2' = \frac{1}{\Gamma(-2\nu)}$ and give

$$H_\nu^2(z) = \frac{1}{\Gamma(-2\nu)}\int_0^{\infty} e^{-t^2-2zt}t^{-2\nu-1}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t^2\right)dt, \quad \Re(\nu) < 0,\ |\arg(z)|<\pi. \tag{2.10}$$

We call the above integral as *the first Lebedev identity* which confirms the result of Lebedev in [2, p. 298, Problem 4].

2.2. Transformations in double integrals

In view of the Lebedev approach for establishing the first identity, we follow his proposition for constructing the double integral

$$H_\nu^2(z) = \frac{1}{\Gamma^2(-\nu)}\int_0^{\infty}\int_0^{\infty} e^{-t^2-s^2-2(t+s)z}\frac{dt\,ds}{(st)^{\nu+1}}, \quad \Re(\nu) < 0, \tag{2.11}$$

and set $s+t=u$ and $\frac{t}{s}=w$ to get

$$H_\nu^2(z)=\frac{1}{\Gamma^2(-\nu)}\int_0^\infty \mathrm{e}^{-u^2-2zu}u^{-2\nu-1}\,\mathrm{d}u\int_0^\infty \mathrm{e}^{\frac{2u^2w}{(1+w)^2}}w^{-\nu-1}(1+w)^{2\nu}\,\mathrm{d}w,\quad \Re(\nu)<0. \tag{2.12}$$

Here, we change the variables $w=\tan^2(\theta)$ and rewrite the inner integral of (2.12) as

$$\begin{aligned}I_1&=\int_0^\infty \mathrm{e}^{\frac{2u^2w}{(1+w)^2}}w^{-\nu-1}(1+w)^{2\nu}\,\mathrm{d}w\\&=2^{2\nu+2}\int_0^{\frac{\pi}{2}}\mathrm{e}^{\frac{1}{2}u^2\sin^2(2\theta)}\sin^{-2\nu-1}(2\theta)\,\mathrm{d}\theta\\&=2^{2\nu+3}\int_0^{\frac{\pi}{4}}\mathrm{e}^{\frac{1}{2}u^2\sin^2(2\theta)}\sin^{-2\nu-1}(2\theta)\,\mathrm{d}\theta.\end{aligned} \tag{2.13}$$

We employ the integral representation (2.4) to get the integral I_1 in the following form (after the suitable transformation)

$$I_1=2^{2\nu+1}\sqrt{\pi}\frac{\Gamma(-\nu)}{\Gamma\left(\frac{1}{2}-\nu\right)}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right). \tag{2.14}$$

The first Lebedev identity is consequently derived after applying the Legendre duplication formula for the gamma functions. By the same procedure, we consider the following double integral

$$H_\nu(z)H_\nu(-z)=\frac{1}{\Gamma^2(-\nu)}\int_0^\infty\int_0^\infty \mathrm{e}^{-t^2-s^2-2(s-t)z}\frac{\mathrm{d}t\,\mathrm{d}s}{(st)^{\nu+1}},\quad \Re(\nu)<0, \tag{2.15}$$

with the change of variables $s-t=u$, $\frac{t}{s}=w$ to get

$$\begin{aligned}H_\nu(z)H_\nu(-z)&=\frac{1}{\Gamma^2(-\nu)}\int_{-\infty}^\infty \mathrm{e}^{-u^2-2zu}u^{-2\nu-2}|u|\,\mathrm{d}u\\&\quad\times\int_0^\infty \mathrm{e}^{-\frac{2u^2w}{(1-w)^2}}w^{-\nu-1}(1-w)^{2\nu}\,\mathrm{d}w,\quad \Re(\nu)<0.\end{aligned} \tag{2.16}$$

At this point, we take into account the inner integral as

$$I_2=\int_0^\infty \mathrm{e}^{-\frac{2u^2w}{(1-w)^2}}w^{-\nu-1}(1-w)^{2\nu}\,\mathrm{d}w=\left[\int_0^1+\int_1^\infty\right]\mathrm{e}^{-\frac{2u^2w}{(1-w)^2}}w^{-\nu-1}(1-w)^{2\nu}\,\mathrm{d}w, \tag{2.17}$$

and set the variables $w=\tanh^2(\theta)$ and $w=\coth^2(\theta)$ for the first and second integrals of (2.17), respectively. Therefore, we have

$$\begin{aligned}I_2&=\int_0^\infty \mathrm{e}^{-\frac{2u^2w}{(1-w)^2}}w^{-\nu-1}(1-w)^{2\nu}\,\mathrm{d}w\\&=2^{2\nu+2}\int_{-\infty}^\infty \mathrm{e}^{-\frac{1}{2}u^2\sinh^2(2\theta)}\sinh^{-2\nu-1}(2\theta)\,\mathrm{d}\theta\end{aligned}$$

$$= 2^{2\nu+3}\int_0^\infty \mathrm{e}^{-\frac{1}{2}u^2\sinh^2(2\theta)}[\sinh^2(2\theta)]^{-\nu-\frac{1}{2}}\,\mathrm{d}\theta. \tag{2.18}$$

We apply the change of variables $\sinh^2(2\theta)=\xi$ and employ the integral representation (2.6) to obtain

$$I_2 = 2^{2\nu+1}\Gamma(-\nu)\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right). \tag{2.19}$$

We finally get *the second Lebedev identity* as follows:

$$\begin{aligned} H_\nu(z)H_\nu(-z) &= \frac{2^{2\nu}}{\Gamma(-\nu)}\int_{-\infty}^\infty \mathrm{e}^{-u^2-2zu}|u|^{-2\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right)\mathrm{d}u \\ &= \frac{2^{2\nu+1}}{\Gamma(-\nu)}\int_0^\infty \mathrm{e}^{-u^2}\cosh(2zu)u^{-2\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right)\mathrm{d}u, \\ &\quad \Re(\nu)<0,\ |\arg(z)|<\pi, \end{aligned} \tag{2.20}$$

or equivalently

$$\begin{aligned} H_\nu(z)H_\nu(-z) &= \mathscr{I}_- + \mathscr{I}_+ \\ &= \frac{2^{2\nu}}{\Gamma(-\nu)}\int_0^\infty \mathrm{e}^{-u^2-2zu}u^{-2\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right)\mathrm{d}u \\ &\quad + \frac{2^{2\nu}}{\Gamma(-\nu)}\int_0^\infty \mathrm{e}^{-u^2+2zu}u^{-2\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right)\mathrm{d}u. \end{aligned} \tag{2.21}$$

Moreover, from the first identity, we can simply get the following integral representation as *the third Lebedev identity* for $H_\nu^2(-z)$

$$H_\nu^2(-z) = \frac{1}{\Gamma(-2\nu)}\int_0^\infty \mathrm{e}^{-t^2+2zt}t^{-2\nu-1}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t^2\right)\mathrm{d}t,\quad \Re(\nu)<0. \tag{2.22}$$

Remark 2.1: We mention that the first integral of (2.21) is interpreted by the alternative integral representations in the next section. For the second integral of (2.21), we take into account

$$\begin{aligned} \Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right) &= \frac{\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma\left(\frac{1}{2}\right)}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right) \\ &\quad + \frac{\Gamma\left(-\frac{1}{2}-\nu\right)}{\Gamma(-\nu)}\frac{1}{2^{\frac{1}{2}+\nu}}u^{1+2\nu}\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}u^2\right),\quad \nu\neq-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots, \end{aligned} \tag{2.23}$$

and reconsider the following representation for $\mathscr{I}_+$ as follows:

$$\begin{aligned} \mathscr{I}_+ &= \frac{2^{2\nu}}{\Gamma(-\nu)}\int_0^\infty \mathrm{e}^{-u^2+2zu}u^{-2\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right)\mathrm{d}u \\ &= \frac{2^{2\nu}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma(-\nu)}\int_0^\infty \mathrm{e}^{-u^2+2zu}u^{-2\nu-1}\Phi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}u^2\right)\mathrm{d}u \end{aligned}$$

$$+\,2^{\nu-\frac{1}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)}{\Gamma^2(-\nu)}\int_0^\infty \mathrm{e}^{-u^2+2zu}\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}u^2\right)\mathrm{d}u. \tag{2.24}$$

In view of the integral representation (2.4) for the function Φ

$$\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}u^2\right)=\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\sqrt{\pi}\Gamma(1+\nu)}\int_0^1 \mathrm{e}^{\frac{1}{2}u^2\xi}\xi^{-\frac{1}{2}}(1-\xi)^\nu\,\mathrm{d}\xi,\quad -1<\Re(\nu)<0, \tag{2.25}$$

the second integral is presented in the following form:

$$\begin{aligned}
&2^{\nu-\frac{1}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)}{\Gamma^2(-\nu)}\int_0^\infty \mathrm{e}^{-u^2+2zu}\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}u^2\right)\mathrm{d}u\\
&=\frac{2^{\nu-\frac{1}{2}}}{\sqrt{\pi}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)}{\Gamma^2(-\nu)}\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+\nu)}\int_0^1 \xi^{-\frac{1}{2}}(1-\xi)^\nu\,\mathrm{d}\xi\\
&\times\int_0^\infty \mathrm{e}^{-(1-\frac{\xi}{2})u^2+2zu}\,\mathrm{d}u.
\end{aligned} \tag{2.26}$$

We now use the following identity

$$\int_0^\infty \mathrm{e}^{-(1-\frac{\xi}{2})u^2+2zu}\,\mathrm{d}u=\frac{1}{2}\sqrt{\frac{\pi}{1-\frac{\xi}{2}}}\,\mathrm{e}^{\frac{z^2}{1-\frac{\xi}{2}}}\,\mathrm{Erfc}\left(\frac{-z}{\sqrt{1-\frac{\xi}{2}}}\right), \tag{2.27}$$

in terms of the complementary error function

$$\mathrm{Erfc}(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-\tau^2}\,\mathrm{d}\tau, \tag{2.28}$$

and for $-1<\Re(\nu)<0$ and $\Re(\nu)\neq-\frac{1}{2}$, we show

$$\begin{aligned}
\mathscr{I}_+=&\frac{2^{2\nu}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma(-\nu)}\Gamma(-2\nu)H_\nu^2(-z)\\
&+2^{\nu-\frac{3}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma^2(-\nu)\Gamma(1+\nu)}\int_0^1\sqrt{\frac{1}{1-\frac{\xi}{2}}}\,\mathrm{e}^{\frac{z^2}{1-\frac{\xi}{2}}}\,\mathrm{Erfc}\left(\frac{-z}{\sqrt{1-\frac{\xi}{2}}}\right)\xi^{-\frac{1}{2}}(1-\xi)^\nu\,\mathrm{d}\xi.
\end{aligned} \tag{2.29}$$

We also note that the following representation holds for $\mathscr{I}_+$

$$\begin{aligned}
\mathscr{I}_+=&\frac{2^{2\nu}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma(-\nu)}\Gamma(-2\nu)H_\nu^2(-z)\\
&+2^{\nu-\frac{1}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)}{\Gamma^2(-\nu)}\int_0^\infty \mathrm{e}^{-u^2+2zu}\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}u^2\right)\mathrm{d}u,
\end{aligned} \tag{2.30}$$

where $\Re(\nu)<0,\nu\neq-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},\ldots$.

3. The integral representations

3.1. The integral representations for first identity

In this section, we consider the integral representation (2.10) in the equivalent form

$$H_\nu^2(z) = \frac{1}{2\Gamma(-2\nu)} \int_0^\infty e^{-t-2z\sqrt{t}} t^{-\nu-1} \Phi\left(-\nu, \frac{1}{2}-\nu; \frac{1}{2}t\right) dt,$$
$$\Re(\nu) < 0,\ |\arg(z)| < \pi, \tag{3.1}$$

and derive some new integral representations for the products of Hermite functions.

Theorem 3.1: *For $|\arg(z)| < \pi$, $|\arg(\zeta)| < \pi$ and $|\arg(z+\zeta)| < \pi$, the following integral representation holds for the product of Hermite functions*

$$H_\nu(z)H_\mu(\zeta) = \frac{\Gamma(-\nu-\mu)}{\Gamma(-\nu)\Gamma(-\mu)} \int_0^{\frac{\pi}{2}} H_{\nu+\mu}\Big(\cos(\theta)z + \sin(\theta)\zeta\Big) \cos^{-\nu-1}(\theta) \sin^{-\mu-1}(\theta)\, d\theta, \tag{3.2}$$

where $\Re(\nu+\mu) < 0$.

Proof: We multiply the relation (1.3) by itself and construct a double integral as follows

$$H_\nu(z)H_\mu(\zeta) = \frac{1}{\Gamma(-\nu)\Gamma(-\mu)} \int_0^\infty \int_0^\infty e^{-(t^2+s^2)-2tz-2s\zeta} \frac{1}{t^{\nu+1}s^{\mu+1}}\, dt\, ds,$$
$$\Re(\nu) < 0,\ \Re(\mu) < 0. \tag{3.3}$$

We transform to the polar coordinates by setting $t = r\cos(\theta)$ and $s = r\sin(\theta)$ and use (1.3), once again, to get the result. ∎

Corollary 3.2: *Setting $\nu = \mu$ in (3.2), multiplying the changed relation (3.2) by itself, and iterating this relation once again, we get an identity for the quartic product of Hermite functions as*

$$\begin{aligned} H_\nu^4(z) &= \frac{\Gamma^2(-2\nu)}{\Gamma^4(-\nu)} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} H_{2\nu}\Big(\cos(\theta_1)z + \sin(\theta_1)z\Big) H_{2\nu}\Big(\cos(\theta_2)z + \sin(\theta_2)z\Big) \\ &\quad \times \cos^{-\nu-1}(\theta_1) \sin^{-\mu-1}(\theta_1) \cos^{-\nu-1}(\theta_2) \sin^{-\mu-1}(\theta_2)\, d\theta_1\, d\theta_2, \\ &= \frac{\Gamma(-4\nu)}{\Gamma^4(-\nu)} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} H_{4\nu}\Big(z\cos(\theta_3)\big(\cos(\theta_1) + \sin(\theta_1)\big) \\ &\quad + z\sin(\theta_3)\big(\cos(\theta_2) + \sin(\theta_2)\big)\Big) \Pi_{j=1}^3 [\cos^{-\nu-1}(\theta_j) \sin^{-\mu-1}(\theta_j)]\, d\theta_1\, d\theta_2\, d\theta_3. \end{aligned} \tag{3.4}$$

This representation can be extended for the product $H_\nu^{2n}(z)$ by the $(2n-1)$-dimensional integral in terms of the function $H_{2n\nu}(z)$.

Theorem 3.3: *For* $|\arg(z)| < \frac{\pi}{2}$ *and* $\Re(\nu) < -\frac{1}{2}$, *the following integral representation holds for the product of Hermite functions*

$$H_\nu^2(z) = \frac{2^{\nu-\frac{1}{2}}}{\sqrt{\pi}\Gamma(-2\nu)} \int_0^\infty \frac{\xi^\nu}{\xi+1} {}_2F_1\left(-\nu, 1; \frac{1}{2}-\nu, \frac{1}{2(\xi+1)}\right) \mathrm{e}^{-\frac{4z^2}{8\xi}} D_{-2\nu-1}\left(\frac{2z}{\sqrt{2\xi}}\right) \mathrm{d}\xi, \tag{3.5}$$

where $D_{-2\nu-1}$ *is the parabolic cylinder function and* ${}_2F_1$ *is the hypergeometric function given by*

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_n (b)_k}{(c)_k} \frac{z^k}{k!}. \tag{3.6}$$

Proof: At first, we consider the following representation for the integrand of (3.1), [14, p. 52, 2.2.1(10)]

$$\mathrm{e}^{-2z\sqrt{t}} t^{-\nu-1} = \frac{1}{2^{-\nu-\frac{1}{2}}\sqrt{\pi}} \int_0^\infty \mathrm{e}^{-\xi t} \xi^\nu \mathrm{e}^{-\frac{4z^2}{8\xi}} D_{-2\nu-1}\left(\frac{2z}{\sqrt{2\xi}}\right) \mathrm{d}\xi, \quad \Re(z^2) > 0, \tag{3.7}$$

and rewrite (3.1) as

$$H_\nu^2(z) = \frac{2^{3\nu+\frac{1}{2}}}{\Gamma(-\nu)\Gamma\left(\frac{1}{2}-\nu\right)} \int_0^\infty \xi^\nu \mathrm{e}^{-\frac{4z^2}{8\xi}} D_{-2\nu-1}\left(\frac{2z}{\sqrt{2\xi}}\right) \mathrm{d}\xi$$
$$\times \int_0^\infty \mathrm{e}^{-t(\xi+1)} \Phi\left(-\nu, \frac{1}{2}-\nu; \frac{1}{2}t\right) \mathrm{d}t, \quad \Re(\nu) < -\frac{1}{2}. \tag{3.8}$$

The condition $\Re(\nu) < -\frac{1}{2}$ is applied for employing the Fubini's theorem and convergence of the outer integral. We now evaluate the inner integral of (3.8) as the Laplace transform of Φ, [15, p. 510, 3.35.1(1)]

$$\int_0^\infty \mathrm{e}^{-t(\xi+1)} \Phi\left(-\nu, \frac{1}{2}-\nu; \frac{1}{2}t\right) \mathrm{d}t = \frac{1}{\xi+1} {}_2F_1\left(-\nu, 1; \frac{1}{2}-\nu, \frac{1}{2(\xi+1)}\right), \tag{3.9}$$

and get the result. ■

Theorem 3.4: *For* $|\arg(z)| < \pi$ *and* $\Re(\nu) < 0$, *the following integral representation holds for the product of Hermite functions*

$$H_\nu^2(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{2^{\nu+\frac{1}{2}}\sqrt{\pi}\Gamma(-\nu)} \int_{\frac{1}{2}}^1 \left(\xi - \frac{1}{2}\right)^{-\frac{1}{2}} (1-\xi)^{-\nu-1} \xi^\nu H_{2\nu}\left(\frac{z}{\sqrt{\xi}}\right) \mathrm{d}\xi, \quad \Re(\nu) < 0. \tag{3.10}$$

Proof: We consider the following representation for the integrand of (3.1), [14, p. 346, 3.33.2(1)]

$$\mathrm{e}^{-t} \Phi\left(-\nu, \frac{1}{2}-\nu; \frac{1}{2}t\right) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{2^{\nu+\frac{1}{2}}\sqrt{\pi}\Gamma(-\nu)} \int_{\frac{1}{2}}^1 \mathrm{e}^{-\xi t} \left(\xi - \frac{1}{2}\right)^{-\frac{1}{2}} (1-\xi)^{-\nu-1} \mathrm{d}\xi, \tag{3.11}$$

and we arrive at

$$H_\nu^2(z) = \frac{1}{\Gamma^2(-\nu)} \int_{\frac{1}{2}}^{1} \left(\xi - \frac{1}{2}\right)^{-\frac{1}{2}} (1-\xi)^{-\nu-1}\,\mathrm{d}\xi \int_0^\infty \mathrm{e}^{-t\xi - 2z\sqrt{t}} t^{-\nu-1}\,\mathrm{d}t, \quad \Re(\nu) < 0. \tag{3.12}$$

We now recall the inner integral of (3.12) as the definition of $H_{2\nu}$

$$\int_0^\infty \mathrm{e}^{-t\xi - 2z\sqrt{t}} t^{-\nu-1}\,\mathrm{d}t = 2\Gamma(-2\nu)\xi^\nu H_{2\nu}\left(\frac{z}{\sqrt{\xi}}\right), \tag{3.13}$$

and get the desired result. ■

Theorem 3.5: *For* $|\arg(z)| < \frac{\pi}{2}$ *and* $\Re(\nu) < 0$, *the following integral representation holds for the product of Hermite functions*

$$H_\nu^2(z) = \frac{\Gamma(-\nu)}{2\Gamma(-2\nu)} \int_1^\infty \frac{z\xi^\nu}{\sqrt{\pi(\xi-1)^3}} \mathrm{e}^{-\frac{z^2}{\xi-1}} {}_2F_1\left(-\nu, -\nu; \frac{1}{2} - \nu, \frac{1}{2\xi}\right) \mathrm{d}\xi. \tag{3.14}$$

Proof: We take into account the following representation for the integrand of (3.1), [14, p. 52, 2.2.1(9)]

$$\mathrm{e}^{-t-2z\sqrt{t}} = \int_1^\infty \mathrm{e}^{-\xi t} \frac{z}{\sqrt{\pi(\xi-1)^3}} \mathrm{e}^{-\frac{z^2}{\xi-1}}\,\mathrm{d}\xi, \quad \Re(z^2) > 0, \tag{3.15}$$

and present the relation (3.1) to

$$H_\nu^2(z) = \frac{1}{2\Gamma(-2\nu)} \int_1^\infty \frac{z}{\sqrt{\pi(\xi-1)^3}} \mathrm{e}^{-\frac{z^2}{\xi-1}}\,\mathrm{d}\xi \times \int_0^\infty \mathrm{e}^{-t\xi} t^{-\nu-1} \Phi\left(-\nu, \frac{1}{2} - \nu; \frac{1}{2}t\right) \mathrm{d}t, \quad \Re(\nu) < 0. \tag{3.16}$$

We now evaluate the inner integral of (3.16) as the Mellin transform of $\mathrm{e}^{-t\xi}\Phi$, [15, p. 510, 3.35.1(2)]

$$\int_0^\infty \mathrm{e}^{-t\xi} t^{-\nu-1} \Phi\left(-\nu, \frac{1}{2} - \nu; \frac{1}{2}t\right) \mathrm{d}t = \frac{\Gamma(-\nu)}{\xi^{-\nu}} {}_2F_1\left(-\nu, -\nu; \frac{1}{2} - \nu, \frac{1}{2\xi}\right), \tag{3.17}$$

and get the result. ■

Theorem 3.6: *For* $|\arg(z)| < \pi$ *and* $\Re(\nu) < 0$, *the following series representation holds for the product of Hermite functions*

$$H_\nu^2(z) = \frac{2^{1+2\nu}}{\Gamma(-\nu)} \sum_{k=0}^\infty \left(-\frac{1}{2}\right)^k \frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma(2k-2\nu)}{k!\,\Gamma\left(k-\frac{\nu}{2}+\frac{1}{2}\right)} H_{2\nu-2k}(\sqrt{2}z). \tag{3.18}$$

Proof: We use the following representation for the integrand of (3.1) in terms of the Bessel function of first kind, [2, p. 278, Problem 12]

$$\Phi\left(-\nu, \frac{1}{2}-\nu; \frac{1}{2}t^2\right) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{2^{\frac{1}{4}+\frac{\nu}{2}}} \mathrm{e}^{\frac{1}{2}t^2} t^{\frac{1}{2}+\nu} \int_0^\infty \mathrm{e}^{-\xi} \xi^{\frac{\nu}{2}-\frac{1}{4}} J_{-\nu-\frac{1}{2}}(t\sqrt{2\xi})\, \mathrm{d}\xi, \tag{3.19}$$

and incorporate into (3.1) to get

$$H_\nu^2(z) = \frac{2^{\frac{3}{4}+\frac{3}{2}\nu}}{\Gamma(-\nu)} \int_0^\infty \mathrm{e}^{-\frac{1}{2}t^2-2zt} t^{-\nu-\frac{1}{2}} J_{-\nu-\frac{1}{2}}(t\sqrt{2\xi}) \int_0^\infty \mathrm{e}^{-\xi} \xi^{\frac{\nu}{2}-\frac{1}{4}}\, \mathrm{d}\xi\, \mathrm{d}t, \quad \Re(\nu) < 0. \tag{3.20}$$

We expand the Bessel function

$$J_\nu(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu+k+1)}, \tag{3.21}$$

into the power series to show

$$\begin{aligned} H_\nu^2(z) = \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\Gamma(-2\nu)\Gamma\left(\frac{1}{2}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{-k}}{k!\Gamma\left(k-\frac{\nu}{2}+\frac{1}{2}\right)} \int_0^\infty \mathrm{e}^{-\frac{1}{2}t^2-2zt} t^{2k-2\nu-1}\, \mathrm{d}t \\ \times \int_0^\infty \mathrm{e}^{-\xi} \xi^{k-\frac{1}{2}}\, \mathrm{d}\xi\, \mathrm{d}t, \quad \Re(\nu) < 0. \end{aligned} \tag{3.22}$$

We apply the relation (1.3) and employ the definition of gamma function for the above integrals to obtain the result. ■

3.2. *The integral representations for second identity*

Here, we consider the first integral of identity (2.21)

$$\mathscr{I}_- = \frac{2^{2\nu-1}}{\Gamma(-\nu)} \int_0^\infty \mathrm{e}^{-t-2z\sqrt{t}} t^{-\nu-1} \Psi\left(-\nu, \frac{1}{2}-\nu; \frac{1}{2}t\right) \mathrm{d}t, \quad \Re(\nu) < 0,\ |\arg(z)| < \pi, \tag{3.23}$$

and state the following theorems to show various representations for the second Lebedev identity (2.21). In all theorems, we incorporate the representation (2.29) for the integral $\mathscr{I}_+$.

Theorem 3.7: *For $|\arg(z)| < \frac{\pi}{2}$ and $-1 < \Re(\nu) < -\frac{1}{2}$, the following integral representation holds for the product of Hermite functions*

$$\begin{aligned} H_\nu(z)H_\nu(-z) &= \frac{1}{2\pi}\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right) H_\nu^2(-z) \\ &+ 2^{\nu-\frac{3}{2}} \frac{\Gamma\left(-\frac{1}{2}-\nu\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+\nu)\Gamma^2(-\nu)} \int_0^1 \sqrt{\frac{1}{1-\frac{\xi}{2}}}\, \mathrm{e}^{\frac{z^2}{1-\frac{\xi}{2}}} \mathrm{Erfc}\left(\frac{-z}{\sqrt{1-\frac{\xi}{2}}}\right) \xi^{-\frac{1}{2}}(1-\xi)^\nu\, \mathrm{d}\xi \end{aligned}$$

$$+\frac{2^{3\nu+\frac{1}{2}}}{\pi}\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(-\nu)}\int_0^\infty \frac{\xi^\nu}{\xi+1}\,{}_2F_1\left(1,-\nu;\frac{3}{2},\frac{\xi+\frac{1}{2}}{\xi+1}\right)\mathrm{e}^{-\frac{4z^2}{8\xi}}D_{-2\nu-1}\left(\frac{2z}{\sqrt{2\xi}}\right)\mathrm{d}\xi. \tag{3.24}$$

Proof: At first, we use the relation (3.7) for the inverse Laplace transform of $\mathrm{e}^{-2z\sqrt{t}}t^{-\nu-1}$ to represent (3.23) as

$$\mathscr{I}_-=\frac{2^{3\nu-\frac{1}{2}}}{\Gamma(-\nu)}\frac{1}{\sqrt{\pi}}\int_0^\infty \xi^\nu\,\mathrm{e}^{-\frac{4z^2}{8\xi}}D_{-2\nu-1}\left(\frac{2z}{\sqrt{2\xi}}\right)\mathrm{d}\xi\int_0^\infty \mathrm{e}^{-t(\xi+1)}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t\right)\mathrm{d}t. \tag{3.25}$$

We now apply the restriction $-\frac{1}{2}<\Re(\nu)<0$ for using the Fubini's theorem and evaluate the inner integral of (3.25) as the Laplace transform of Ψ [15, p. 518, 3.36.1(1)]

$$\int_0^\infty \mathrm{e}^{-t(\xi+1)}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t\right)\mathrm{d}t=\frac{1}{\xi+1}\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma\left(\frac{3}{2}\right)}\,{}_2F_1\left(1,-\nu;\frac{3}{2},\frac{\xi+\frac{1}{2}}{\xi+1}\right), \tag{3.26}$$

and get the result after recalling (2.29). ■

Theorem 3.8: *For $|\arg(z)|<\frac{\pi}{2}$, $-1<\Re(\nu)<0$ and $\Re(\nu)\neq-\frac{1}{2}$, the following integral representation holds for the product of Hermite functions*

$$\begin{aligned}H_\nu(z)H_\nu(-z)&=\frac{1}{2\pi}\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)H_\nu^2(-z)\\&+2^{\nu-\frac{3}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+\nu)\Gamma^2(-\nu)}\int_0^1\sqrt{\frac{1}{1-\frac{\xi}{2}}}\,\mathrm{e}^{\frac{z^2}{1-\frac{\xi}{2}}}\mathrm{Erfc}\left(\frac{-z}{\sqrt{1-\frac{\xi}{2}}}\right)\xi^{-\frac{1}{2}}(1-\xi)^\nu\,\mathrm{d}\xi\\&+\frac{2^{\nu-1}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\nu\right)}\int_1^\infty\frac{z}{\sqrt{\pi(\xi-1)^3}}\mathrm{e}^{-\frac{z^2}{\xi-1}}\,{}_2F_1\left(-\nu,\frac{1}{2};\frac{1}{2}-\nu,\frac{\frac{1}{2}-\xi}{\xi}\right)\mathrm{d}\xi.\end{aligned} \tag{3.27}$$

Proof: We consider (3.15) as the inverse Laplace transform of $\mathrm{e}^{-t-2z\sqrt{t}}$ and substitute into the integrand of (3.23) to obtain

$$\mathscr{I}_-=\frac{2^{2\nu-1}}{\Gamma(-\nu)}\int_1^\infty\frac{z}{\sqrt{\pi(\xi-1)^3}}\mathrm{e}^{-\frac{z^2}{\xi-1}}\mathrm{d}\xi\int_0^\infty \mathrm{e}^{-t\xi}t^{-\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t\right)\mathrm{d}t. \tag{3.28}$$

We now evaluate the inner integral of (3.28) as the Mellin transform of $\mathrm{e}^{-t\xi}\,\Psi$, [15, p. 518, 3.36.1(2)]

$$\int_0^\infty \mathrm{e}^{-t\xi}t^{-\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t\right)\mathrm{d}t=\frac{1}{2^\nu}\frac{\Gamma(-\nu)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-\nu\right)}\,{}_2F_1\left(-\nu,\frac{1}{2};\frac{1}{2}-\nu,\frac{\frac{1}{2}-\xi}{\xi}\right), \tag{3.29}$$

and get the result. ■

Theorem 3.9: *For* $|\arg(z)| < \pi$ *and* $-\frac{1}{2} < \Re(\nu) < 0$, *the following integral representation holds for the product of Hermite functions*

$$H_\nu(z)H_\nu(-z) = \frac{1}{2\pi}\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)H_\nu^2(-z)$$
$$+ 2^{\nu-\frac{3}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+\nu)\Gamma^2(-\nu)}\int_0^1\sqrt{\frac{1}{1-\frac{\xi}{2}}}\,\mathrm{e}^{\frac{z^2}{1-\frac{\xi}{2}}}\,\mathrm{Erfc}\left(\frac{-z}{\sqrt{1-\frac{\xi}{2}}}\right)\xi^{-\frac{1}{2}}(1-\xi)^\nu\,\mathrm{d}\xi$$
$$+ \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{2^{\nu+\frac{5}{2}}\sqrt{\pi}\,\Gamma(-\nu)}\int_0^\infty \xi^{-\nu-1}\left(\frac{1}{2}+\xi\right)^{-\frac{1}{2}}(1+\xi)^\nu H_{2\nu}\left(\frac{z}{\sqrt{\xi+1}}\right)\mathrm{d}\xi. \tag{3.30}$$

Proof: We use the following representation as the inverse Laplace transform of $\Psi(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t)$, [14, p. 351, 3.34.1(1)]

$$\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t\right) = \frac{1}{\Gamma(-\nu)2^{\frac{1}{2}+\nu}}\int_0^\infty \mathrm{e}^{-t\xi}\xi^{-\nu-1}\left(\frac{1}{2}+\xi\right)^{-\frac{1}{2}}\mathrm{d}\xi, \tag{3.31}$$

and substitute into (3.23) to get

$$\begin{aligned}\mathscr{I}_- &= \frac{2^{\nu-\frac{3}{2}}}{\Gamma^2(-\nu)}\int_0^\infty \xi^{-\nu-1}\left(\frac{1}{2}+\xi\right)^{-\frac{1}{2}}\mathrm{d}\xi\int_0^\infty \mathrm{e}^{-t^2(\xi+1)}\,\mathrm{e}^{-2zt}t^{-2\nu-1}\,\mathrm{d}t\\ &= \frac{2^{\nu-\frac{3}{2}}\Gamma(-2\nu)}{\Gamma^2(-\nu)}\int_0^\infty \xi^{-\nu-1}\left(\frac{1}{2}+\xi\right)^{-\frac{1}{2}}(1+\xi)^\nu H_{2\nu}\left(\frac{z}{\sqrt{\xi+1}}\right)\mathrm{d}\xi,\\ &\quad -\frac{1}{2} < \Re(\nu) < 0.\end{aligned} \tag{3.32}$$

The condition $-\frac{1}{2} < \Re(\nu) < 0$ is used for employing the Fubini's theorem and convergence of the outer integral.The result is finally obtained after considering the representation (2.29) for $\mathscr{I}_+$. ■

Theorem 3.10: *For* $|\arg(z)| < \pi$ *and* $\Re(\nu) < -1, \nu \neq -\frac{3}{2}, -\frac{5}{2}, \ldots$, *the following integral representation holds for the product of Hermite functions*

$$H_\nu(z)H_\nu(-z) = \frac{1}{2\pi}\Gamma\left(\frac{1}{2}+\nu\right)\Gamma\left(\frac{1}{2}-\nu\right)H_\nu^2(-z)$$
$$+ 2^{\nu-\frac{1}{2}}\frac{\Gamma\left(-\frac{1}{2}-\nu\right)}{\Gamma^2(-\nu)}\int_0^\infty \mathrm{e}^{-\xi^2+2z\xi}\,\Phi\left(\frac{1}{2},\frac{3}{2}+\nu;\frac{1}{2}\xi^2\right)\mathrm{d}\xi \tag{3.33}$$
$$+ \frac{2^{\nu-1}}{\Gamma(-\nu)}\int_0^\infty\left[\frac{1}{\xi+1} - z\sqrt{\frac{\pi}{(\xi+1)^3}}\,\mathrm{e}^{\frac{z^2}{\xi+1}}\,\mathrm{Erfc}\left(\frac{z}{\sqrt{\xi+1}}\right)\right]{}_2F_1\left(-\nu,\frac{1}{2};1,-2\xi\right)\mathrm{d}\xi. \tag{3.34}$$

Proof: At first, we consider the inverse Laplace transform of $t^{-\nu-1}\Psi(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t)$, [14, p. 351, 3.34.1(2)]

$$t^{-\nu-1}\Psi\left(-\nu,\frac{1}{2}-\nu;\frac{1}{2}t\right) = \frac{1}{2^\nu}\int_0^\infty \mathrm{e}^{-t\xi}\,{}_2F_1\left(-\nu,\frac{1}{2};1,-2\xi\right)\mathrm{d}\xi, \tag{3.35}$$

and substitute into integrand of (3.23) to obtain

$$\mathscr{I}_- = \frac{2^{\nu-1}}{\Gamma(-\nu)} \int_0^\infty {}_2F_1\left(-\nu, \frac{1}{2}; 1, -2\xi\right) \mathrm{d}\xi \int_0^\infty \mathrm{e}^{-t(\xi+1)}\, \mathrm{e}^{-2z\sqrt{t}}\, \mathrm{d}t, \quad \Re(\nu) < -1. \tag{3.36}$$

The condition $\Re(\nu) < -1$ is applied for employing the Fubini's theorem and convergence of the outer integral. We now evaluate the inner integral of (3.36) as the Laplace transform of $\mathrm{e}^{-t\xi}\Psi$, [15, p. 30, 2.2.1(15)]

$$\int_0^\infty \mathrm{e}^{-t(\xi+1)}\, \mathrm{e}^{-2z\sqrt{t}}\, \mathrm{d}t = \frac{1}{\xi+1} - z\sqrt{\frac{\pi}{(\xi+1)^3}}\, \mathrm{e}^{\frac{z^2}{\xi+1}}\, \mathrm{Erfc}\left(\frac{z}{\sqrt{\xi+1}}\right), \tag{3.37}$$

and take into account the relation (2.30) to get the result. ■

Remark 3.1: In view of the relation (2.30) as the substitution of (2.29) for the integral $\mathscr{I}_+$, the result of Theorem 3.7 can be replaced by the restriction $\Re(\nu) < -\frac{1}{2}, \nu \neq -\frac{3}{2}, -\frac{5}{2}, \ldots$. Also, the result of Theorem 3.8 can be replaced by the restriction $\Re(\nu) < 0, \nu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

The financial support of this work was provided by the Shahrekord University [grant number 98GRD30M1328].

ORCID

Alireza Ansari http://orcid.org/0000-0003-4387-3779

References

[1] Namias V. The fractional order Fourier transform and its application to quantum mechanics. IMA J Appl Math. 1980;25(3):241–265.
[2] Lebedev NN. Special functions & their applications. New York: Dover Publications; 1972.
[3] Veestraeten D. Some integral representations and limits for (products of) the parabolic cylinder function. Integral Transforms Spec Funct. 2016;27(1):64–77.
[4] Veestraeten D. An integral representation for the product of parabolic cylinder functions. Integral Transforms Spec Funct. 2017;28(1):15–21.
[5] Veestraeten D. An alternative integral representation for the product of two parabolic cylinder functions. Integral Transforms Spec Funct. 2017;28(12):915–922.
[6] Veestraeten D. On the inverse transform of Laplace transforms that contain (products of) the parabolic cylinder function. Integral Transforms Spec Funct. 2015;26(11):859–871.
[7] Glasser ML. An integral representation for the product of two parabolic cylinder functions having unrelated arguments. Integral Transforms Spec Funct. 2015;26(10):825–828.
[8] Malyshev C. A Nicholson-type integral for the product of two parabolic cylinder functions $D_\nu(x)D_{-\nu}(-x)$ at $\Re\nu < 0$. Integral Transforms Spec Funct. 2003;14(2):139–148.
[9] Nasri R. Product of parabolic cylinder functions involving Laplace transforms of confluent hypergeometric functions. Integral Transforms Spec Funct. 2016;27(3):181–196.

[10] Kamke E. Differentialgleichungen: Lösungsmethoden und Lösungen, I, Gewöhnliche Differentialgleichungen. Leipzig: B. G. Teubner; 1948.
[11] Ansari A, Eshaghi S. Lipschitz-Hankel identities and integral representations of toroidal functions. Integral Transforms Spec Funct. 2020. Available from: doi:10.1080/10652469.2020.1855637.
[12] Ansari A, Eshaghi S. Macdonald's identities and integral representations of products of Airy functions. Integral Transforms Spec Funct. 2020;31(9):744–757.
[13] Ansari A, Masomi MR. Some results involving integral transforms of Airy functions. Integral Transforms Spec Funct. 2015;26(7):539–549.
[14] Prudnikov AP, Brychkov YuA, Marichev OI. Integrals and series, Vol. 5, inverse laplace transforms. London: Gordon and Breach; 1993.
[15] Prudnikov AP, Brychkov YuA, Marichev OI. Integrals and series, Vol. 4, direct laplace transforms. London: Gordon and Breach; 1993.