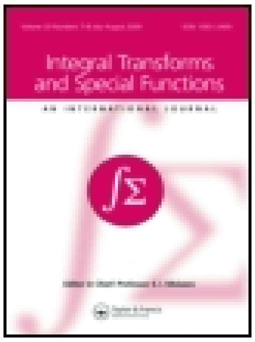

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Lipschitz–Hankel identities and integral representations of toroidal functions

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Lipschitz–Hankel identities and integral representations of toroidal functions

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ABSTRACT
In this paper, using the Lipschitz–Hankel identities we obtain some new integral representations for the toroidal functions in terms of the elementary, Bessel, parabolic cylinder and hypergeometric functions. Manipulating the integrands of Lipschitz–Hankel identities with several integral representations lead us to present the results. In this sense, we also derive some integral representations for the products of toroidal functions.

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1. Introduction

The Lipschitz–Hankel identities (integrals) are the Laplace integral transforms of the product of one or more Bessel functions in the form [1, p. 386]

$$\mathscr{L}\{t^{\nu}\mathfrak{B}_{\mu}(t);s\} = \int_0^{\infty} \mathrm{e}^{-st} t^{\nu}\mathfrak{B}_{\mu}(t)\,\mathrm{d}t, \tag{1.1}$$

where $\mathfrak{B}_{\mu}$ is a Bessel or modified Bessel function of order μ (or their product). These integrals are mostly expressed in terms of the hypergeometric and associated Legendre functions. The associated Legendre functions P_{ν}^{μ} and Q_{ν}^{μ} are the solutions of the associated Legendre differential equation

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\nu(\nu+1) - \frac{\mu^2}{1-x^2}\right)y = 0. \tag{1.2}$$

We note that in view of the three notations of these functions, we here use the standard Hobson definition of P_{ν}^{μ} and Q_{ν}^{μ} [2]. See also the notation of Olver [3] and Barnes [4]. When $\nu = n - \frac{1}{2}, n = 0, 1, 2, \ldots, \mu \in \mathbb{R}$ and $x \in (1, \infty)$ the solutions of (1.2) are known as toroidal or ring functions. These functions have the fundamental roles in boundary value problems in toroidal coordinates, see[5,6]. Motivated by Lipschitz–Hankel integral (1.1),

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and taking into account the modified Bessel functions of first kind I_ν

$$I_\nu(\xi) = \sum_{k=0}^{\infty} \frac{\xi^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu + k + 1)}, \tag{1.3}$$

and the modified Bessel functions of second kind K_ν (Macdonald function)

$$K_\nu(\xi) = \frac{\pi}{2} \csc(\pi \nu)[I_{-\nu}(\xi) - I_\nu(\xi)], \quad \nu \notin \mathbb{Z}, \tag{1.4}$$

we consider the following integral representations for the associated Legendre functions [1, p. 387]

$$\begin{aligned} P_\nu^{-\mu}(\cosh(\alpha)) &= \frac{1}{\Gamma(\nu + \mu + 1)} \\ &\times \int_0^\infty e^{-t\cosh(\alpha)} I_\mu(t \sinh(\alpha)) t^\nu \, dt, \quad \Re(\nu + \mu) > -1, \ \alpha > 0, \end{aligned} \tag{1.5}$$

$$\begin{aligned} Q_\nu^\mu(\cosh(\alpha)) &= \frac{e^{\mu\pi i}}{\Gamma(\nu - \mu + 1)} \\ &\times \int_0^\infty e^{-t\cosh(\alpha)} K_\mu(t \sinh(\alpha)) t^\nu \, dt, \quad \Re(\nu + 1) > |\Re(\mu)|, \ \alpha > 0, \end{aligned} \tag{1.6}$$

and also the following pair form which can be derived from Whipple's formula [3, Exercise 12.2] and [1, p. 388]

$$\begin{aligned} Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh(\alpha)) &= e^{(\nu-\frac{1}{2})\pi i} \sinh^{\nu-\frac{1}{2}}(\alpha) \sqrt{\frac{\pi}{2}} \\ &\times \int_0^\infty e^{-t\cosh(\alpha)} I_\mu(t) t^{\nu-1} \, dt, \quad \Re(\nu + \mu) > 0, \quad \alpha > 0, \end{aligned} \tag{1.7}$$

$$\begin{aligned} P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) &= \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)\sqrt{\frac{2}{\pi}}}{\Gamma(\nu - \mu)\Gamma(\nu + \mu)} \\ &\times \int_0^\infty e^{-t\cosh(\alpha)} K_\mu(t) t^{\nu-1} \, dt, \quad \Re(\nu) > |\Re(\mu)|, \quad \alpha > 0. \end{aligned} \tag{1.8}$$

We mention that the above identities have been considered as the emendations of the Lipschitz–Hankel integrals presented by Gradshteyn and Ryzhik [7, p. 705, 6.628(4–7)]. See [8] for studying these integrals and their emendations. For the various integral representations for the associated Legendre functions, the interested reader is referred to [9, Section 8]. Because, the introduced Lipschitz–Hankel identities (1.5)–(1.8) can be expressed as the Laplace, Mellin and Kontorovich–Lebedev transforms, we employ the operational calculi of these transforms along with the different integral representations of integrands (1.5)–(1.8) to show some new integral representations for the toroidal functions. We use the tables of integral transforms to construct our results in the next sections. This technique is a simple and effective approach to derive new representations for many special functions in the applied mathematics and physics, for example see [10,11]. In this sense, we organize the paper to get new integral representations for the toroidal functions in terms of the elementary, Bessel, parabolic cylinder functions and also their products in Section 2. Section 3 is devoted to the integral representations involving the quadratic products of the associated Legendre functions and particularly the products of toroidal functions.

2. Integral representations of toroidal functions

2.1. Integral representations in terms of the elementary functions

Theorem 2.1: *For $\Re(\mu) > -\frac{1}{2}$ and $\Re(\nu+1) > |\Re(\mu)|$, the following integral representations hold for $Q_\nu^\mu(\cosh(\alpha))$:*

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{-\mu}\, e^{\mu\pi i} \sinh^{-\mu}(\alpha) \Gamma\left[\begin{smallmatrix}\mu+\nu+1\\ \mu+1\end{smallmatrix}\right]}{\Gamma(\nu-\mu+1)\cosh^{\mu+\nu+1}(\alpha)} \times \int_0^\infty \frac{\tau^{2\mu+1}}{\tau^2+\sinh^2(\alpha)} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) d\tau, \tag{2.1}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{-\mu}\, e^{\mu\pi i} \Gamma\left[\begin{smallmatrix}\mu+\nu+1\\ \mu+1\end{smallmatrix}\right] \cosh^{-\mu-\nu-1}(\alpha)}{\Gamma(\nu-\mu+1)\sinh^{\mu-1}(\alpha)\sinh(\eta\sinh(\alpha))} \times \int_0^\infty \frac{\tau^{2\mu}\sin(\eta\tau)}{\tau^2+\sinh^2(\alpha)} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) d\tau, \tag{2.2}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{-\mu}\, e^{\mu\pi i}(\alpha) \Gamma\left[\begin{smallmatrix}\mu+\nu+1\\ \mu+1\end{smallmatrix}\right] \cosh^{-\mu-\nu-1}(\alpha)}{\Gamma(\nu-\mu+1)\sinh^{\mu}\cosh(\eta\sinh(\alpha))} \times \int_0^\infty \frac{\tau^{2\mu+1}\cos(\eta\tau)}{\tau^2+\sinh^2(\alpha)} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) d\tau, \tag{2.3}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{\sqrt{\pi}\, e^{\mu\pi i}\Gamma(\nu+a+\frac{1}{2})}{4\Gamma(\nu-\mu+1)\Gamma(a)\sinh^{\frac{1}{2}-a}(\alpha)} \times \int_0^\infty \frac{\tau^{a-1}}{(\cosh(\alpha)+(\tau+1)\sinh(\alpha))^{\nu+a+\frac{1}{2}}} \times {}_2F_1\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; a; -\frac{\tau}{2}\right) d\tau. \tag{2.4}$$

Proof: Using the following integral representations for the Macdonald function [7, p. 917, 8.432(5,8,9)], [7, p. 352, 3.391]

$$K_\mu(xz) = \frac{\Gamma(\mu+\frac{1}{2})(2z)^\mu}{x^\mu\sqrt{\pi}} \int_0^\infty \frac{\cos(x\tau)}{(\tau^2+z^2)^{\mu+\frac{1}{2}}} d\tau, \quad \Re(\mu) > -\frac{1}{2}, \; x > 0, \tag{2.5}$$

$$K_\mu(xz) = \sqrt{\frac{\pi}{2z}} \frac{x^\mu\, \mathrm{e}^{-xz}}{\Gamma(\mu+\frac{1}{2})} \int_0^\infty \mathrm{e}^{-x\tau} \tau^{\mu-\frac{1}{2}} \left(1+\frac{\tau}{2z}\right)^{\mu-\frac{1}{2}} \mathrm{d}\tau, \quad \Re(\mu) > -\frac{1}{2},\ x>0, \tag{2.6}$$

$$K_\mu(xz) = \frac{\sqrt{\pi}}{\Gamma(\mu+\frac{1}{2})} \left(\frac{x}{2z}\right)^\mu \int_0^\infty \frac{1}{\sqrt{\tau^2+z^2}}\, \mathrm{e}^{-x\sqrt{\tau^2+z^2}} \tau^{2\mu}\, \mathrm{d}\tau, \quad \Re(\mu) > -\frac{1}{2},\ x>0, \tag{2.7}$$

$$K_\mu(xz) = \frac{x\,\mathrm{e}^{-zx}}{\mu 2^{\mu+1} z^\mu} \int_0^\infty ((\sqrt{\tau+2z}+\sqrt{\tau})^{2\mu} - (\sqrt{\tau+2z}-\sqrt{\tau})^{2\mu}) e^{-x\tau}\, \mathrm{d}\tau, \quad |\arg z| < \pi,\ \Re(x) > 0, \tag{2.8}$$

and setting $x = t$ and $z = \sinh(\alpha)$ in (2.5)–(2.8) we get the results (2.1)–(2.4), respectively. It should be noted that for deriving relation (2.1), we employ the identity [12, p. 75, 2.4.2(1)]

$$\int_0^\infty \mathrm{e}^{-t\cosh(\alpha)} \cos(t\tau) t^{\nu-\mu}\, \mathrm{d}t = \frac{\Gamma(\nu-\mu+1)}{(\cosh^2(\alpha)+\tau^2)^{(\nu-\mu+1)/2}} \cos\left((\nu-\mu+1)\tan^{-1}\left(\frac{\tau}{\cosh(\alpha)}\right)\right), \tag{2.9}$$

and use the following well-known identity for establishing the relations (2.2)–(2.4)

$$\int_0^\infty \mathrm{e}^{-ts} t^p\, \mathrm{d}t = \frac{\Gamma(p+1)}{s^{p+1}}, \quad p = \nu \pm \mu. \tag{2.10}$$

■

Theorem 2.2: *For $\Re(\mu) > -\frac{1}{2}$ and $\Re(\nu) > |\Re(\mu)|$, the following integral representations hold for $P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha))$:*

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)}{\Gamma(\nu+\mu)} \frac{\Gamma(\mu+\frac{1}{2})2^{\mu+\frac{1}{2}}}{\pi} \times \int_0^\infty \frac{\cos\left((\nu-\mu)\tan^{-1}(\frac{\tau}{\cosh(\alpha)})\right)}{(\cosh^2(\alpha)+\tau^2)^{(\nu-\mu)/2}} \frac{1}{(\tau^2+1)^{\mu+\frac{1}{2}}}\, \mathrm{d}\tau, \tag{2.11}$$

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)}{\Gamma(\nu-\mu)\Gamma(\mu+\frac{1}{2})} \times \int_0^\infty \frac{\tau^{\mu-\frac{1}{2}}(1+\frac{\tau}{2})^{\mu-\frac{1}{2}}}{(\tau+\cosh(\alpha)+1)^{\nu+\mu}}\, \mathrm{d}\tau, \tag{2.12}$$

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)}{2^{\mu-\frac{1}{2}}\Gamma(\mu+\frac{1}{2})\Gamma(\nu-\mu)} \times \int_0^\infty \frac{\tau^{2\mu}}{\sqrt{\tau^2+1}} \frac{1}{\left(\cosh(\alpha)+\sqrt{\tau^2+1}\right)^{\nu+\mu}}\, \mathrm{d}\tau, \tag{2.13}$$

$$P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) = \frac{2^{-\mu-\frac{1}{2}}\sinh^{\nu-\frac{1}{2}}(\alpha)\Gamma(\nu+1)}{\mu\sqrt{\pi}\Gamma(\nu-\mu)\Gamma(\nu+\mu)} \times \int_0^\infty \frac{(\sqrt{\tau+2}+\sqrt{\tau})^{2\mu}-(\sqrt{\tau+2}-\sqrt{\tau})^{2\mu}}{(\cosh(\alpha)+\tau+1)^{\nu+1}}\,\mathrm{d}\tau. \tag{2.14}$$

Proof: The proof of this theorem is similar to the proof of Theorem 2.1. The only substitutions are $z = 1$ and $x = t$ in (2.5)–(2.8). ■

2.2. Integral representations in terms of the Bessel functions

Theorem 2.3: *For $\Re(\nu+1) > |\Re(\mu)|$, the following integral representations hold for $Q_\nu^\mu(\cosh(\alpha))$. For $|\Re(\mu)| < 1$, we have*

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{1-\frac{\nu}{2}}\,\mathrm{e}^{\mu\pi\mathrm{i}}\sinh^{-\frac{\nu}{2}-1}(\alpha)}{\Gamma(\nu-\mu+1)\,\mathrm{e}^{\frac{\alpha\nu}{2}}} \times \int_0^\infty \tau^{\nu+1}\left(I_\mu(\tau)+I_{-\mu}(\tau)\right)K_\mu(\tau)K_\nu\left(\sqrt{\frac{2\mathrm{e}^\alpha}{\sinh(\alpha)}}\tau\right)\mathrm{d}\tau, \tag{2.15}$$

and for $|\Re(\mu)| < \frac{1}{2}$, we have

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{8\mathrm{e}^{\mu\pi\mathrm{i}}\cos(\mu\pi)\,\mathrm{e}^{-\alpha(\frac{\nu}{2}-\frac{1}{4})}}{\sqrt{\pi}\Gamma(\nu-\mu+1)}\int_0^\infty \tau^{\nu+\frac{1}{2}}K_{2\mu}(\tau\sqrt{8\sinh(\alpha)})K_{\nu+\frac{1}{2}}(2\tau\,\mathrm{e}^{\frac{\alpha}{2}})\,\mathrm{d}\tau. \tag{2.16}$$

Proof: For the right-hand side of (1.6), we apply the following integral representations for the Macdonald function [7, p. 708, 6.634], [7, p. 699, 6.618(3)]

$$K_\mu(x) = \frac{\mathrm{e}^{-x}}{x}\int_0^\infty \tau\,\mathrm{e}^{-\frac{\tau^2}{2x}}\left(I_\mu(\tau)+I_{-\mu}(\tau)\right)K_\mu(\tau)\,\mathrm{d}\tau, \quad \Re(x)>0, \quad -1<\Re(\mu)<1, \tag{2.17}$$

$$K_\mu\left(\frac{z^2}{8x}\right) = \frac{4\sqrt{x}}{\sqrt{\pi}\,\sec(\mu\pi)}\,\mathrm{e}^{-\frac{z^2}{8x}}\int_0^\infty \mathrm{e}^{-x\tau^2}K_{2\mu}(z\tau)\,\mathrm{d}\tau, \quad \Re(x)>0, \quad |\Re(\mu)|<\frac{1}{2}, \tag{2.18}$$

and set $x = t\sinh(\alpha)$ in (2.17) and $x = \frac{1}{t}$, $z = \sqrt{8\sinh(\alpha)}$ in (2.18). By employing the following Laplace transform [12, p. 31, 2.2.2(1)]

$$\int_0^\infty \mathrm{e}^{-ts}t^p\,\mathrm{e}^{-\frac{a}{t}}\,\mathrm{d}t = 2\left(\frac{a}{s}\right)^{(p+1)/2}K_{p+1}(2\sqrt{as}), \quad \Re(a)>0, \quad \Re(s)>0, \tag{2.19}$$

we get the results. ■

Theorem 2.4: *For $\Re(\nu) > |\Re(\mu)|$, the following integral representations hold for $P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha))$. For $|\Re(\mu)| < 1$, we have*

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{2^{2-\frac{\nu}{2}}\sinh^{\nu-\frac{1}{2}}(\alpha)}{\sqrt{\pi}\Gamma(\nu-\mu)\Gamma(\nu+\mu)(\cosh(\alpha)+1)^{\frac{\nu-1}{2}}} \times \int_0^\infty \tau^{\nu}(I_\mu(\tau)+I_{-\mu}(\tau))K_\mu(\tau)K_{\nu-1}(\sqrt{2(\cosh(\alpha)+1)}\tau)\,\mathrm{d}\tau, \tag{2.20}$$

and for $|\Re(\mu)| < \frac{1}{2}$, we have

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{8\sqrt{2}\sinh^{\nu-\frac{1}{2}}(\alpha)\cos(\mu\pi)}{\pi\Gamma(\nu-\mu)\Gamma(\nu+\mu)\sqrt{(\cosh(\alpha)+1)^{\nu-\frac{1}{2}}}} \times \int_0^\infty \tau^{\nu-\frac{1}{2}}K_{2\mu}(\sqrt{8}\tau)K_{\nu-\frac{1}{2}}(2\tau\sqrt{\cosh(\alpha)+1})\,\mathrm{d}\tau. \tag{2.21}$$

Proof: The proof of this theorem is similar to the proof of previous theorem for the Macdonald function in the right-hand side of (1.8). We set $x = t$ in (2.17) and $x = \frac{1}{t}$, $z = \sqrt{8}$ in (2.18) to get the results. ∎

2.3. Integral representations in terms of the parabolic cylinder function

Theorem 2.5: *For $\Re(\nu+1) > |\Re(\mu)|$, the following integral representation holds for $Q^\mu_\nu(\cosh(\alpha))$:*

$$Q^\mu_\nu(\cosh(\alpha)) = 2^{\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}}\sinh^\mu(\alpha)\,\mathrm{e}^{\mu\pi\mathrm{i}} \times \int_0^\infty \mathrm{e}^{-\sinh^2(\alpha)\tau+\frac{1}{2}\tau\cosh^2(\alpha)}\tau^{\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}}D_{\mu-\nu-1}(\sqrt{2\tau}\cosh(\alpha))\,\mathrm{d}\tau, \tag{2.22}$$

where D_{-s} is the parabolic cylinder function given by [7, p. 1028, 9.241(2)]

$$\int_0^\infty t^{s-1}\mathrm{e}^{-\eta t^2-\lambda t}\,\mathrm{d}t = (2\eta)^{-\frac{s}{2}}\Gamma(s)\,\mathrm{e}^{\frac{\lambda^2}{8\eta}}D_{-s}\left(\frac{\lambda}{\sqrt{2\eta}}\right), \quad \Re(s) > 0. \tag{2.23}$$

Proof: First we employ the relation (2.19) once again and set $x = \frac{t^2}{4}$ and $z = \sinh^2(\alpha)$ in the following integral:

$$\int_0^\infty \mathrm{e}^{-z\tau-\frac{x}{\tau}}\tau^{\mu-1}\,\mathrm{d}\tau = 2\left(\frac{x}{z}\right)^{\frac{\mu}{2}}K_\mu(2\sqrt{xz}), \quad \Re(z) > 0,\ \Re(x) > 0, \tag{2.24}$$

to get

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{\mu-1}\sinh^\mu(\alpha)\,\mathrm{e}^{\mu\pi \mathrm{i}}}{\Gamma(\nu-\mu+1)}\int_0^\infty \tau^{\mu-1}\,\mathrm{e}^{-\sinh^2(\alpha)\tau}\,\mathrm{d}\tau \times \int_0^\infty \mathrm{e}^{-\frac{t^2}{4\tau}}\,\mathrm{e}^{-t\cosh(\alpha)}t^{\nu-\mu}\,\mathrm{d}t \tag{2.25}$$

The result is obtained after considering the following representation for the parabolic cylinder function. ■

Remark 2.6: By taking into account the integrand of (2.22) in the case $\nu = \mu$ and considering [7, p. 336, 3.322(2)]

$$D_{-1}(z) = \sqrt{\frac{\pi}{2}}\,\mathrm{e}^{\frac{z^2}{4}}\,\mathrm{Erfc}\left(\frac{z}{\sqrt{2}}\right), \tag{2.26}$$

we get a representation in terms of the complementary error function as follows:

$$Q_\nu^\nu(\cosh(\alpha)) = 2^{\nu-1}\sqrt{\pi}\sinh^\nu(\alpha)\,\mathrm{e}^{\nu\pi \mathrm{i}}\int_0^\infty \tau^{\nu-\frac{1}{2}}\,\mathrm{e}^\tau\,\mathrm{Erfc}\left(\cosh(\alpha)\sqrt{\tau}\right)\mathrm{d}\tau, \tag{2.27}$$

where

$$\mathrm{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi. \tag{2.28}$$

Theorem 2.7: *For* $\Re(\nu) > |\Re(\mu)|$, *the following integral representation holds for* $P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))$:

$$P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) = \frac{2^{\frac{1}{2}\nu+\frac{1}{2}\mu-\frac{1}{2}}\sinh^{\nu-\frac{1}{2}}(\alpha)}{\sqrt{\pi}\,\Gamma(\nu+\mu)} \times \int_0^\infty \mathrm{e}^{-\frac{\tau}{2}(2-\cosh^2(\alpha))}\tau^{\frac{1}{2}\nu+\frac{1}{2}\mu-1}D_{-\nu+\mu}(\sqrt{2\tau}\cosh(\alpha))\,\mathrm{d}\tau. \tag{2.29}$$

Proof: We consider (1.8) and substitute $x = \frac{t^2}{4}$ and $z = 1$ into (2.24). We proceed with the same procedure of previous proof to obtain

$$P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) = \frac{2^{\mu-1}\sinh^{\nu-\frac{1}{2}}(\alpha)\sqrt{\frac{2}{\pi}}}{\Gamma(\nu-\mu)\Gamma(\nu+\mu)} \times \int_0^\infty \mathrm{e}^{-\tau}\tau^{\mu-1}\,\mathrm{d}\tau\int_0^\infty \mathrm{e}^{-t\cosh(\alpha)}\,\mathrm{e}^{-\frac{t^2}{4\tau}}t^{\nu-\mu-1}\,\mathrm{d}t. \tag{2.30}$$

The result is derived after the implementation of the integral (2.23) for the parabolic cylinder function. ■

Theorem 2.8: *For $\Re(\nu+1) > |\Re(\mu)|$, the following integral representations hold for* $Q_\nu^\mu(\cosh(\alpha))$:

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{\frac{\nu-\mu-1}{2}}\, \mathrm{e}^{\mu\pi \mathrm{i}}\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)\sinh^{\nu+1}(\alpha)} \times \int_0^\infty \frac{\mathrm{e}^{-\frac{\tau}{2}(2-\coth^2(\alpha))}}{\tau^{\frac{\mu-\nu+1}{2}}} D_{-\nu-\mu-1}(\sqrt{2\tau}\coth(\alpha))\,\mathrm{d}\tau, \tag{2.31}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{\mathrm{e}^{\mu\pi \mathrm{i}}\Gamma(\nu+\mu+1)}{2\Gamma(\nu-\mu+1)\sinh^{\frac{\nu-\mu+1}{2}}(\alpha)} \times \int_0^\infty \frac{\mathrm{e}^{-\frac{\tau}{4}(2\sinh(\alpha)-\coth^2(\alpha))}}{\tau^{\frac{\mu-\nu+1}{2}}} D_{-\nu-\mu-1}(\sqrt{\tau}\coth(\alpha))\,\mathrm{d}\tau. \tag{2.32}$$

Proof: We use the following integral representations for the Macdonald function [7, p. 917, 8.432(6,7)]

$$K_\mu(x) = \frac{x^\mu}{2^{\mu+1}} \int_0^\infty \frac{\mathrm{e}^{-\tau-\frac{x^2}{4\tau}}}{\tau^{\mu+1}}\,\mathrm{d}\tau, \quad |\arg x| < \frac{\pi}{2}, \quad \Re(x^2) > 0, \tag{2.33}$$

$$K_\mu(xz) = \frac{x^\mu}{2} \int_0^\infty \tau^{-\mu-1}\,\mathrm{e}^{-\frac{z}{2}(\tau+\frac{x^2}{\tau})}\,\mathrm{d}\tau, \quad |\arg x| < \frac{\pi}{4}, \tag{2.34}$$

and set $x = t\sinh(\alpha)$ in (2.33), and $x = t$, $z = \sinh(\alpha)$ in (2.34). The result is derived after recalling the integral (2.23) for the parabolic cylinder function. ■

Theorem 2.9: *For $\Re(\nu) > |\Re(\mu)|$, the following integral representations hold for* $P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))$:

$$P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) = \frac{2^{\frac{\nu}{2}-\frac{\mu}{2}-\frac{1}{2}}\sinh^{\nu-\frac{1}{2}}(\alpha)}{\sqrt{\pi}\Gamma(\nu-\mu)} \int_0^\infty \frac{\mathrm{e}^{-\frac{\tau}{2}(1-\sinh^2(\alpha))}}{\tau^{\frac{\mu}{2}-\frac{\nu}{2}+1}} D_{-\nu-\mu}(\sqrt{2\tau}\cosh(\alpha))\,\mathrm{d}\tau, \tag{2.35}$$

$$P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)}{\sqrt{2\pi}\Gamma(\nu-\mu)} \int_0^\infty \frac{\mathrm{e}^{-\frac{\tau}{4}(1-\sinh^2(\alpha))}}{\tau^{\frac{\mu}{2}-\frac{\nu}{2}+1}} D_{-\nu-\mu}(\sqrt{\tau}\cosh(\alpha))\,\mathrm{d}\tau. \tag{2.36}$$

Proof: The proof of this theorem is similar to the proof of previous theorem. The only substitutions are $x = t$ in (2.33), and $x = t$, $z = 1$ in (2.34). ■

2.4. Integral representations in terms of the hypergeometric function

In this subsection, we intend to represent some integral representations for the toroidal functions in terms of the hypergeometric function

$${}_2F_1(a,b;c;z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{2.37}$$

where $(\cdot)_n$ is the Pochhammer symbol.

Theorem 2.10: *For $\Re(\nu+1) > |\Re(\mu)|$, the following integral representations hold for* $Q_\nu^\mu(\cosh(\alpha))$:

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{-\mu}\, e^{\mu\pi i}\sinh^{-\mu}(\alpha)\Gamma\left[\begin{matrix}\mu+\nu+1\\ \mu+1\end{matrix}\right]}{\Gamma(\nu-\mu+1)\cosh^{\mu+\nu+1}(\alpha)} \times \int_0^\infty \frac{\tau^{2\mu+1}}{\tau^2+\sinh^2(\alpha)} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) d\tau, \tag{2.38}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{-\mu}\, e^{\mu\pi i}\Gamma\left[\begin{matrix}\mu+\nu+1\\ \mu+1\end{matrix}\right]\cosh^{-\mu-\nu-1}(\alpha)}{\Gamma(\nu-\mu+1)\sinh^{\mu-1}(\alpha)\sinh(\eta\sinh(\alpha))} \times \int_0^\infty \frac{\tau^{2\mu}\sin(\eta\tau)}{\tau^2+\sinh^2(\alpha)} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) d\tau, \tag{2.39}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{-\mu}\, e^{\mu\pi i}(\alpha)\Gamma\left[\begin{matrix}\mu+\nu+1\\ \mu+1\end{matrix}\right]\cosh^{-\mu-\nu-1}(\alpha)}{\Gamma(\nu-\mu+1)\sinh^{\mu}\cosh(\eta\sinh(\alpha))} \times \int_0^\infty \frac{\tau^{2\mu+1}\cos(\eta\tau)}{\tau^2+\sinh^2(\alpha)} \times {}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) d\tau, \tag{2.40}$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{\sqrt{\pi}\, e^{\mu\pi i}\Gamma(\nu+a+\frac{1}{2})}{4\Gamma(\nu-\mu+1)\Gamma(a)\sinh^{\frac{1}{2}-a}(\alpha)} \times \int_0^\infty \frac{\tau^{a-1}}{(\cosh(\alpha)+(\tau+1)\sinh(\alpha))^{\nu+a+\frac{1}{2}}} \times {}_2F_1\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; a; -\frac{\tau}{2}\right) d\tau. \tag{2.41}$$

Proof: We recall the Bessel function of first kind

$$J_\nu(\xi) = \sum_{k=0}^\infty \frac{(-1)^k \xi^{2k+\nu}}{2^{2k+\nu} k!\Gamma(\nu+k+1)}, \tag{2.42}$$

and consider the following integral representation for the Macdonald function [7, p. 679, 6.566(2), 735, 814, 819], [7, p. 735, 6.718(1,2) 814, 819], [7, p. 814, 7.522(2)] and [7, p. 819, 7.542(16)]

$$K_\mu(xz) = z^{-\mu}\int_0^\infty \frac{\tau^{\mu+1}}{\tau^2+z^2} J_\mu(x\tau)\, d\tau, \quad x>0,\ \Re(z)>0,\ -1<\Re(\mu)<\frac{3}{2}, \tag{2.43}$$

$$K_\mu(xz) = \frac{z^{1-\mu}}{\sinh(\eta z)} \int_0^\infty \frac{\tau^\mu}{\tau^2+z^2} \sin(\eta\tau) J_\mu(x\tau)\, \mathrm{d}\tau,$$
$$0 < \eta \le x,\ \Re(z) > 0,\ -1 < \Re(\mu) < \frac{3}{2}, \tag{2.44}$$

$$K_\mu(xz) = \frac{z^{-\mu}}{\cosh(\eta z)} \int_0^\infty \frac{\tau^{\mu+1}}{\tau^2+z^2} \cos(\eta\tau) J_\mu(x\tau)\, \mathrm{d}\tau,$$
$$0 < \eta \le x,\ \Re(z) > 0,\ -1 < \Re(\mu) < \frac{1}{2}, \tag{2.45}$$

$$K_\mu(x) = \frac{\mathrm{e}^{-x}\sqrt{\pi}}{2^a \Gamma(a)} (2x)^{a-\frac{1}{2}} \int_0^\infty \tau^{a-1}\, \mathrm{e}^{-x\tau}\, {}_2F_1\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; a; -\frac{\tau}{2}\right) \mathrm{d}\tau,$$
$$\Re(x) > 0,\ \Re(a) > 0. \tag{2.46}$$

We apply the integral (1.6) once again and set $x = t, z = \sinh(\alpha)$ in (2.43)–(2.45). If we use the following Laplace transform for the Bessel function of first kind [12, p. 256, 3.12.1(2)]

$$\int_0^\infty \mathrm{e}^{-ts} t^p J_\mu(at)\, \mathrm{d}t = \frac{a^\mu}{2^\mu s^{\mu+p+1}} \Gamma\begin{bmatrix} \mu+p+1 \\ \mu+1 \end{bmatrix}$$
$$\times {}_2F_1\left(\frac{\mu+p+1}{2}, \frac{\mu+p+2}{2}; \mu+1; -\frac{a^2}{s^2}\right), \quad \Re(p+\mu) > -1, \tag{2.47}$$

then, we get the relations (2.38)–(2.40), respectively. We note that the notation $\Gamma[{}^{(a_m)}_{(b_n)}]$ is the ratio of the gamma functions as

$$\Gamma\begin{bmatrix} (a_m) \\ (b_n) \end{bmatrix} = \Gamma\begin{bmatrix} a_1, \ldots, a_m \\ b_1, \ldots, b_n \end{bmatrix} = \frac{\Pi_{j=1}^m \Gamma(a_j)}{\Pi_{j=1}^n \Gamma(b_j)}.$$

Also, if we set $x = t\sinh(\alpha)$ in (2.46) and use the identity (2.10), the integral representation (2.41) is obtained. ∎

Theorem 2.11: *For $\Re(\nu) > |\Re(\mu)|$, the following integral representations hold for $P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha))$:*

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{2^{\frac{1}{2}-\mu} \sinh^{\nu-\frac{1}{2}}(\alpha)}{\sqrt{\pi}\,\Gamma(\nu-\mu)\Gamma(1+\mu)\cosh^{\mu+\nu}(\alpha)}$$
$$\times \int_0^\infty \frac{\tau^{2\mu+1}}{\tau^2+1} {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) \mathrm{d}\tau, \tag{2.48}$$

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{2^{\frac{1}{2}-\mu} \sinh^{\nu-\frac{1}{2}}(\alpha) \cosh^{-\mu-\nu}(\alpha)}{\sqrt{\pi}\,\Gamma(\nu-\mu)\Gamma(1+\mu)\sinh(\eta)}$$
$$\times \int_0^\infty \frac{\tau^{2\mu}\sin(\eta\tau)}{\tau^2+1} {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right) \tag{2.49}$$

$$\mathrm{d}\tau, P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{2^{1-\mu}\sinh^{\nu-\frac{1}{2}}(\alpha)\cosh^{-\mu-\nu}(\alpha)}{\sqrt{\pi}\Gamma(\nu-\mu)\Gamma(1+\mu)\cosh(\eta)} \times \int_0^\infty \frac{\tau^{2\mu+1}\cos(\eta\tau)}{\tau^2+1} \times {}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}; \mu+1; -\frac{\tau^2}{\cosh^2(\alpha)}\right)\mathrm{d}\tau, \tag{2.50}$$

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)\Gamma(\nu+a-\frac{1}{2})}{\Gamma(\nu-\mu)\Gamma(\nu+\mu)\Gamma(a)} \times \int_0^\infty \frac{\tau^{a-1}}{(\cosh(\alpha)+\tau+1)^{\nu+a-\frac{1}{2}}}{}_2F_1\left(\frac{1}{2}+\mu, \frac{1}{2}-\mu; a; -\frac{\tau}{2}\right)\mathrm{d}\tau. \tag{2.51}$$

Proof: The proof of this theorem is similar to the proof of previous theorem for the relation (1.8). Here, we substitute $x = t$, $z = 1$ into (2.43)–(2.45), and $x = t$ into (2.46). ■

2.5. Integral representations in terms of the products of the parabolic cylinder and Bessel functions

Theorem 2.12: *The following integral representations hold for* $Q^\mu_\nu(\cosh(\alpha))$ *and* $P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}$ $(\cosh(\alpha))$. *For* $\Re(\nu+1) > |\Re(\mu)|$, *we have*

$$Q^\mu_\nu(\cosh(\alpha)) = \frac{e^{\mu\pi i}\Gamma(\nu+2)}{2^{\frac{\nu}{2}+2}\sqrt{\pi}\Gamma(\nu-\mu+1)}\int_0^\infty \frac{e^{\frac{1}{8\tau}}}{\tau^{\frac{\nu+3}{2}}} \times K_{\frac{\mu}{2}}\left(\frac{\sinh^2(\alpha)}{8\tau}\right)D_{-\nu-2}\left(\frac{\cosh(\alpha)}{\sqrt{2\tau}}\right)\mathrm{d}\tau, \tag{2.52}$$

and for $\Re(\nu) > |\Re(\mu)|$, *we have*

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)\Gamma(\nu+1)}{2^{\frac{\nu}{2}+1}\pi\Gamma(\nu+\mu)\Gamma(\nu-\mu)}\int_0^\infty \frac{e^{\frac{\sinh^2(\alpha)}{8\tau}}}{\tau^{\frac{\nu}{2}+1}}K_{\frac{\mu}{2}}\left(\frac{1}{8\tau}\right) \times D_{-\nu-1}\left(\frac{\cosh(\alpha)}{\sqrt{2\tau}}\right)\mathrm{d}\tau. \tag{2.53}$$

Proof: By substituting $x = t^2$, $z = \sinh(\alpha)$ and $x = t^2$, $z = 1$ into the following relation [7, p. 713, 6.654]:

$$K_\mu(z\sqrt{x}) = \frac{\sqrt{x}}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{z^2}{8\tau}-x\tau}}{\sqrt{\tau}}K_{\frac{\mu}{2}}\left(\frac{z^2}{8\tau}\right)\mathrm{d}\tau, \tag{2.54}$$

and applying (2.23) as the definition of the parabolic cylinder function, the results are obtained, respectively. ■

The next theorems are related to the Buschman theorem for the Laplace transforms of the composite functions [13, p. 76, Table A.]. We use the Laplace transform of function $t^\nu f(t^2)$ and establish new representations for the toroidal functions.

Theorem 2.13: *For $|\Re(\mu)| < 2$, the following integral representations hold for $Q_\nu^\mu(\cosh(\alpha))$ and $P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha))$. For $\Re(\nu+1) > |\Re(\mu)|$, we have*

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{\pi\, \mathrm{e}^{\mu\pi \mathrm{i}} \sinh(\alpha)\csc(\frac{\mu\pi}{2})}{2^{\frac{\nu}{2}+4}\Gamma(\nu-\mu+1)} \int_0^\infty \frac{\mathrm{e}^{\frac{1}{8u}}}{u^{\frac{\nu}{2}+3}} D_\nu\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \left[K_{\frac{\mu+1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right) - K_{\frac{\mu-1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right)\right] \tag{2.55}$$

and for $\Re(\nu) > |\Re(\mu)|$, we have

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sqrt{\pi}\sinh^{\nu-\frac{1}{2}}(\alpha)\sinh(\alpha)\csc(\frac{\mu\pi}{2})}{2^{\frac{\nu}{2}+3}\Gamma(\nu-\mu)\Gamma(\nu+\mu)} \int_0^\infty \frac{\mathrm{e}^{\frac{-1}{8u}}}{u^{\frac{\nu}{2}+\frac{5}{2}}} D_{\nu-1}\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \left[K_{\frac{\mu+1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right) - K_{\frac{\mu-1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right)\right] \mathrm{d}u. \tag{2.56}$$

Proof: From the integral representation (1.6), we can rewrite

$$\begin{aligned} Q_\nu^\mu(\cosh(\alpha)) &= \frac{\mathrm{e}^{\mu\pi\mathrm{i}}}{\Gamma(\nu-\mu+1)}\int_0^\infty \mathrm{e}^{-t\cosh(\alpha)} K_\mu(t\sinh(\alpha)) t^\nu\, \mathrm{d}t \\ &= \frac{\mathrm{e}^{\mu\pi\mathrm{i}}}{\Gamma(\nu-\mu+1)} \mathscr{L}\left\{t^\nu f(t^2); \cosh(\alpha)\right\}, \end{aligned} \tag{2.57}$$

which by applying the following Laplace transform [13, p. 76, Table A(A-1)]:

$$\mathscr{L}\{t^\nu f(t^2); s\} = \frac{2^{-1-\frac{1}{2}\nu}}{\sqrt{\pi}} \int_0^\infty u^{-\frac{\nu}{2}-\frac{1}{2}}\, \mathrm{e}^{-\frac{s^2}{8u}} D_\nu\left(\frac{s}{\sqrt{2u}}\right) \mathscr{L}\{f(t); u\}\, \mathrm{d}u, \tag{2.58}$$

leads to

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{\mathrm{e}^{\mu\pi\mathrm{i}} 2^{-1-\frac{1}{2}\nu}}{\sqrt{\pi}\Gamma(\nu-\mu+1)} \int_0^\infty u^{-\frac{\nu}{2}-\frac{1}{2}}\, \mathrm{e}^{-\frac{\cosh^2(\alpha)}{8u}} D_\nu\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \mathscr{L}\{K_\mu(\sqrt{t}\sinh(\alpha)); u\}\, \mathrm{d}u. \tag{2.59}$$

At this point, we use the following Laplace transform for $f(t) = K_\mu(\sqrt{t}\sinh(\alpha))$ [12, p. 352, 3.16.2(1)]

$$\mathscr{L}\{K_\mu(a\sqrt{t}); s\} = \left(\frac{\pi}{s}\right)^{\frac{3}{2}} \frac{a}{8s}\csc\left(\mu\frac{\pi}{2}\right) \mathrm{e}^{\frac{a^2}{8s}} \times \left[K_{\frac{\mu+1}{2}}\left(\frac{a^2}{8s}\right) - K_{\frac{\mu-1}{2}}\left(\frac{a^2}{8s}\right)\right], \quad |\Re(\mu)| < 2,\ \mu \neq 0, \pm 1, \tag{2.60}$$

and deduce (2.55). Similarly, from the integral representation (1.8), the relation (2.56) is obtained. ■

Remark 2.14: If we fix $\mu = 0$ in (2.55) and use the following Laplace transform [12, p. 352, 3.16.2(2)]

$$\mathscr{L}\{K_0(a\sqrt{t}); s\} = -\frac{1}{s}\, \mathrm{e}^{\frac{a^2}{4s}}\, \mathrm{Ei}\left(\frac{a^2}{4s}\right), \tag{2.61}$$

in terms of the exponential integral

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^t}{t}\, \mathrm{d}t, \tag{2.62}$$

then, the following integral representation is obtained for $\Re(\nu) > -1$

$$Q_\nu(\cosh(\alpha)) = \frac{-2^{-\frac{\nu}{2}-1}}{\sqrt{\pi}\,\Gamma(\nu+1)} \int_0^\infty \frac{\mathrm{e}^{-\frac{1}{8u}(1-\sinh^2(\alpha))}}{u^{\frac{\nu}{2}+\frac{3}{2}}} D_\nu\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \mathrm{Ei}\left(\frac{\sinh^2(\alpha)}{4u}\right) \mathrm{d}u. \tag{2.63}$$

Theorem 2.15: *For $\Re(\mu) > -2$, the following integral representations hold for $P_\nu^{-\mu}$ $(\cosh(\alpha))$ and $Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh(\alpha))$. For $\Re(\nu+1) > |\Re(\mu)|$, we have*

$$P_\nu^{-\mu}(\cosh(\alpha)) = \frac{\sinh(\alpha)}{2^{\frac{\nu}{2}+3}\Gamma(\nu+\mu+1)} \int_0^\infty \frac{\mathrm{e}^{-\frac{1}{8u}}}{u^{\frac{\nu}{2}+2}} D_\nu\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \left[I_{\frac{\mu+1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right) + I_{\frac{\mu-1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right)\right] \mathrm{d}u, \tag{2.64}$$

and for $\Re(\nu) > |\Re(\mu)|$, we have

$$Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sqrt{\pi}}{2^{\frac{\nu}{2}+3}}\, \mathrm{e}^{(\nu-\frac{1}{2})\pi \mathrm{i}} \sinh^{\nu+\frac{3}{2}}(\alpha) \int_0^\infty \frac{\mathrm{e}^{-\frac{1}{8u}}}{u^{\frac{\nu}{2}+\frac{3}{2}}} D_{\nu-\frac{1}{2}}\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \left[I_{\frac{\mu+1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right) + I_{\frac{\mu-1}{2}}\left(\frac{\sinh^2(\alpha)}{8u}\right)\right] \mathrm{d}u. \tag{2.65}$$

Proof: According to the integral representation (1.5) and applying the Laplace transform (2.58), we can write

$$\begin{aligned} P_\nu^{-\mu}(\cosh(\alpha)) &= \frac{1}{\Gamma(\nu+\mu+1)} \int_0^\infty \mathrm{e}^{-t\cosh(\alpha)} I_\mu\left(t\sinh(\alpha)\right) t^\nu\, \mathrm{d}t \\ &= \frac{1}{\Gamma(\nu+\mu+1)} \mathscr{L}\left\{t^\nu f(t^2); \cosh(\alpha)\right\} \end{aligned}$$

$$
= \frac{2^{-1-\frac{1}{2}\nu}}{\sqrt{\pi}\Gamma(\nu+\mu+1)} \int_0^\infty u^{-\frac{\nu}{2}-\frac{1}{2}} \mathrm{e}^{-\frac{\cosh^2(\alpha)}{8u}} D_\nu\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right) \times \mathscr{L}\{I_\mu(\sqrt{t}\sinh(\alpha)); u\}\,\mathrm{d}u. \tag{2.66}
$$

By applying the following Laplace transform for the function $f(t) = I_\mu(\sqrt{t}\sinh(\alpha))$ [12, p.317, 3.15.2(1)]:

$$
\mathscr{L}\{I_\mu(a\sqrt{t}); s\} = \frac{a\sqrt{\pi}}{4s^{\frac{3}{2}}}\,\mathrm{e}^{\frac{a^2}{8s}}\left[I_{\frac{\mu+1}{2}}\left(\frac{a^2}{8s}\right) + I_{\frac{\mu-1}{2}}\left(\frac{a^2}{8s}\right)\right], \quad \Re(\mu) > -2,\ \mu \neq 0, \pm 1, \tag{2.67}
$$

the relation (2.64) is obtained. Similarly, by starting from the integral representation (1.7), the relation (2.65) is established. ■

Remark 2.16: If we fix $\mu = 0$ in (2.64), then by employing the following Laplace transform [12, p.317, 3.15.2(2)]

$$
\mathscr{L}\{I_0(a\sqrt{t}); s\} = \frac{1}{s}\,\mathrm{e}^{\frac{a^2}{4s}}, \tag{2.68}
$$

we get

$$
P_\nu(\cosh(\alpha)) = \frac{2^{-\frac{\nu}{2}-1}}{\sqrt{\pi}\Gamma(\nu+1)} \int_0^\infty \frac{\mathrm{e}^{-\frac{1}{8u}(1-\sinh^2(\alpha))}}{u^{\frac{1}{2}\nu+\frac{3}{2}}} D_\nu\left(\frac{\cosh(\alpha)}{\sqrt{2u}}\right)\mathrm{d}u, \quad \Re(\nu) > -1. \tag{2.69}
$$

2.6. Integral representations in terms of the products of hypergeometric and Bessel functions

Theorem 2.17: *For $\Re(\nu+1) > |\Re(\mu)|$, the following integral representations hold for $Q_\nu^\mu(\cosh(\alpha))$:*

$$
\begin{aligned}
Q_\nu^\mu(\cosh(\alpha)) = \frac{\mathrm{e}^{\mu\pi\mathrm{i}}}{\Gamma(\nu-\mu+1)} \int_0^\infty & \tau^{\mu+1} J_\mu(\tau\sinh(\alpha)) \\
& \times \left(\frac{1}{2} B\left(\frac{\nu-\mu+1}{2}, \frac{\mu-\nu+1}{2}\right)\tau^{\nu-\mu-1}\right. \\
& \times {}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{1}{2}; \frac{\nu-\mu+1}{2}; \frac{-\tau^2\cosh^2(\alpha)}{4}\right) \\
& - \frac{\cosh(\alpha)}{2} B\left(\frac{\nu-\mu+2}{2}, \frac{\mu-\nu}{2}\right)\tau^{\nu-\mu} \\
& \times {}_2F_1\left(\frac{\nu-\mu+2}{2}, \frac{3}{2}; \frac{\nu-\mu+2}{2}; \frac{-\tau^2\cosh^2(\alpha)}{4}\right) \\
& \left. + \frac{\Gamma(\nu-\mu-1)}{\cosh^{\nu-\mu-1}(\alpha)}\right)\mathrm{d}\tau,
\end{aligned} \tag{2.70}
$$

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{\beta-\mu-1}\, e^{\mu\pi i}\Gamma(\beta)\eta^{1-\beta}\Gamma\begin{bmatrix}\mu+\nu+1\\ \mu+1\end{bmatrix}}{\Gamma(\nu-\mu+1)\sinh^\mu(\alpha)\cosh^{\mu+\nu+1}(\alpha)}$$
$$\times \int_0^\infty \frac{\tau^{2\mu+1}{}_2F_1\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \mu+1; \frac{-\tau^2}{\cosh^2(\alpha)}\right)}{\sqrt{(\tau^2+\sinh^2(\alpha))^{\beta+1}}}$$
$$\times J_{\beta-1}\left(\eta\sqrt{\tau^2+\sinh^2(\alpha)}\right) d\tau, \tag{2.71}$$

where B is the beta function given by

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}. \tag{2.72}$$

Proof: We use the following integral representation for the Macdonald function [7, p. 679, 6.566(2)]:

$$K_\mu(xz) = x^{-\mu}\int_0^\infty \frac{\tau^{\mu+1}}{x^2+\tau^2}J_\mu(z\tau)\, d\tau, \quad z>0, \quad \Re(x)>0, \quad -1<\Re(\mu)<\frac{3}{2}, \tag{2.73}$$

and set $x = t$, $z = \sinh(\alpha)$ in the right-hand side of (1.6). We apply the Laplace transform of $t^\eta(t^2+x^2)^\lambda$ [12, p. 20, 2.1.4(2)] for the obtained double integral and get (2.70). Further, by setting $x = t$, $z = \sinh(\alpha)$ in the following integral representation [7, p. 693, 6.596(4)]:

$$K_\mu(xz) = \left(\frac{\eta}{2}\right)^{1-\beta}\frac{\Gamma(\beta)}{z^\mu}\int_0^\infty \tau^{\mu+1}J_\mu(x\tau)\frac{J_{\beta-1}\left(\eta\sqrt{\tau^2+z^2}\right)}{\sqrt{(\tau^2+z^2)^{\beta+1}}}\, d\tau,$$
$$\eta < x,\ \Re(\beta+2) > \Re(\mu) > -1, \tag{2.74}$$

and applying (2.47), we get (2.71). ■

Theorem 2.18: *For* $\Re(\nu) > |\Re(\mu)|$, *the following integral representations hold for* $P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha))$:

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)\sqrt{2/\pi}}{\Gamma(\nu-\mu)\Gamma(\nu+\mu)}\int_0^\infty \tau^{\mu+1}J_\mu(\tau)$$
$$\times\left(\frac{1}{2}B(\frac{\nu-\mu}{2}, \frac{\mu-\nu+2}{2})\tau^{\nu-\mu-2}\right.$$
$$\times {}_2F_1\left(\frac{\nu-\mu}{2}, \frac{1}{2}, \frac{\nu-\mu}{2}; \frac{-\tau^2\cosh^2(\alpha)}{4}\right)$$
$$-\frac{\cosh(\alpha)}{2}B\left(\frac{\nu-\mu+1}{2}, \frac{\mu-\nu+1}{2}\right)\tau^{\nu-\mu-1}$$

$$\times {}_2F_1\left(\frac{\nu-\mu+1}{2}, \frac{3}{2}, \frac{\nu-\mu+1}{2}; \frac{-\tau^2\cosh^2(\alpha)}{4}\right)$$
$$+\frac{\Gamma(\nu-\mu-2)}{\cosh^{\nu-\mu-2}(\alpha)}\Bigg) \,\mathrm{d}\tau, \tag{2.75}$$

$$P^{\frac{1}{2}-\nu}_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{2^{\beta-\mu-\frac{1}{2}}\sinh^{\nu-\frac{1}{2}}(\alpha)\Gamma(\beta)\eta^{1-\beta}}{\sqrt{\pi}\Gamma(\nu-\mu)\Gamma(1+\mu)\cosh^{\mu+\nu}(\alpha)}$$
$$\times \int_0^\infty \frac{\tau^{2\mu+1}{}_2F_1\left(\frac{\mu+\nu}{2}, \frac{\mu+\nu+1}{2}, \mu+1; \frac{-\tau^2}{\cosh^2(\alpha)}\right)}{\sqrt{(\tau^2+1)^{\beta+1}}} J_{\beta-1}\left(\eta\sqrt{\tau^2+1}\right) \mathrm{d}\tau. \tag{2.76}$$

Proof: The results are proved similar to the proof of Theorem 2.17. The only change is the putting $x = t$, $z = 1$ in (2.73) and (2.74). ■

2.7. Integral representations in terms of the product of confluent hypergeometric and parabolic cylinder functions

In this subsection, we intend to represent integral representation for the toroidal functions in terms of the confluent hypergeometric function

$$ {}_1F_1(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}, \tag{2.77}$$

where $(\cdot)_n$ is the Pochhammer symbol.

Theorem 2.19: *For* $\Re(\nu+1) > |\Re(\mu)|$, *the following integral representation holds for* $Q^{\mu}_{\nu}(\cosh(\alpha))$:

$$Q^{\mu}_{\nu}(\cosh(\alpha)) = \frac{\sinh^{\mu}(\alpha)(\mathrm{e}^{\alpha})^{-\mu-\nu-1}\Gamma\left[\begin{matrix}\mu+\nu+1\\ \mu+1\end{matrix}\right]}{\pi 2^{\frac{1}{2}-2\mu}\,\mathrm{e}^{-\mu\pi\mathrm{i}}\Gamma(2\mu)\Gamma(\nu-\mu+1)\sin(\mu\pi)}$$
$$\times \int_0^\infty \mathrm{e}^{\frac{1}{4}\tau^2} D_{2\mu-1}(\tau)\tau^{2\mu}{}_1F_1\left(\mu+\nu+1; \mu+1; -\frac{\tau^2\sinh(\alpha)}{\mathrm{e}^{\alpha}}\right) \mathrm{d}\tau. \tag{2.78}$$

Proof: For the relation (1.6), we employ the following relation [7, p. 845, 7.752(2)] and fix $x = 2\sqrt{t\sinh(\alpha)}$

$$K_\mu\left(\frac{1}{4}x^2\right) = \frac{2^{\mu-\frac{1}{2}}x^{\mu}}{\pi\sin(\mu\pi)\Gamma(2\mu)}\,\mathrm{e}^{-\frac{1}{4}x^2}\int_0^\infty \tau^{\mu}\,\mathrm{e}^{-\frac{1}{4}\tau^2}D_{2\mu-1}(\tau)J_\mu(x\tau)\,\mathrm{d}\tau,$$
$$x>0,\ -\frac{1}{2} < \Re(\mu) < \frac{1}{2}, \tag{2.79}$$

to get

$$Q_\nu^\mu(\cosh(\alpha)) = \frac{2^{2\mu-\frac{1}{2}}\, e^{\mu\pi i} \sinh^{\frac{\mu}{2}}(\alpha)}{\pi\Gamma(2\mu)\Gamma(\nu-\mu+1)\sin(\mu\pi)} \int_0^\infty e^{\frac{1}{4}\tau^2} D_{2\mu-1}(\tau)\tau^\mu \, d\tau \times \int_0^\infty e^{-t(\cosh(\alpha)+\sinh(\alpha))} t^{\nu+\frac{\mu}{2}} J_\mu(2\tau\sqrt{t\sinh(\alpha)})dt. \tag{2.80}$$

At this point, we consider the following identity for the above double integral [12, p. 262, 3.12.3(5)]:

$$\int_0^\infty e^{-ts} t^p J_\mu(a\sqrt{t})\, dt = \Gamma\begin{bmatrix} p+\frac{\mu}{2}+1 \\ \mu+1 \end{bmatrix} \frac{(a/2)^\mu}{s^{p+\frac{\mu}{2}+1}} {}_1F_1\left(p+\frac{\mu}{2}+1; \mu+1; -\frac{a^2}{4s}\right), \quad \Re(2p+\mu) > -2,\ \Re(s) > 0,\ |\arg(a)| < \pi, \tag{2.81}$$

and obtain the result. ■

In order to show the integral representation for $P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))$ in terms of the confluent hypergeometric and parabolic cylinder functions, we use (1.8) once again and substitute $x = 2\sqrt{t}$ into the integral representation (2.79). Therefore, we can get the following theorem.

Theorem 2.20: *For $\Re(\nu) > |\Re(\mu)|$, the following integral representation holds for $P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))$:*

$$P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha)) = \frac{\sinh^{\nu-\frac{1}{2}}(\alpha)(\cosh(\alpha)+1)^{-\mu-\nu}}{\pi^{\frac{3}{2}} 2^{-2\mu}\Gamma(2\mu)\Gamma(\nu-\mu)\Gamma(1+\mu)\sin(\mu\pi)} \times \int_0^\infty e^{\frac{1}{4}\tau^2} D_{2\mu-1}(\tau)\tau^{2\mu} {}_1F_1\left(\mu+\nu; \mu+1; -\frac{\tau^2}{\cosh(\alpha)+1}\right) d\tau. \tag{2.82}$$

3. Integral representations for the products of associated Legendre functions

In this section, using the integral transforms of the convolution products for the Laplace, Mellin and Kontorovich-Lebedev transforms [14]

$$\mathscr{L}\left\{\int_0^t f(t-\tau)g(\tau)\, d\tau; s\right\} = \int_0^\infty e^{-st} \int_0^t f(t-\tau)g(\tau)\, d\tau\, dt = \mathscr{L}\{f(t); s\}\mathscr{L}\{g(t); s\}, \tag{3.1}$$

$$\mathscr{M}\left\{\int_0^\infty f\left(\frac{t}{\tau}\right) g(\tau)\frac{d\tau}{\tau}; s\right\} = \int_0^\infty t^{s-1} \int_0^\infty f\left(\frac{t}{\tau}\right) g(\tau)\frac{d\tau}{\tau}\, dt = \mathscr{M}\{f(t); s\}\mathscr{M}\{g(t); s\}, \tag{3.2}$$

$$\begin{aligned}
&\mathscr{K}\mathscr{L}\left\{\frac{1}{2t}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}(\frac{v}{u}+\frac{u}{v})t-\frac{uv}{2t}}f(u)g(v)\,du\,dv;\tau\right\}\\
&=\int_0^\infty\frac{K_{i\tau}(t)}{2t}\int_0^\infty\int_0^\infty e^{-\frac{1}{2}(\frac{v}{u}+\frac{u}{v})t-\frac{uv}{2t}}f(u)g(v)\,du\,dv\,dt\\
&=\mathscr{K}\mathscr{L}\{f(t);\tau\}\mathscr{K}\mathscr{L}\{g(t);\tau\},
\end{aligned}\tag{3.3}$$

we get new integral representations for the products of associated Legendre functions.

Theorem 3.1: *For $\Re(\nu+1)>|\Re(\mu)|$, the following integral representation holds:*

$$\begin{aligned}
&\int_0^\infty \Gamma^2(\nu-i\mu+1)\,e^{2\mu\pi}\,[Q_\nu^{i\mu}(\cosh(\alpha))]^2\,d\mu\\
&=\frac{\pi^{\frac{3}{2}}\Gamma\begin{bmatrix}\nu+1,\nu+1,2\nu+2\\ 2\nu+\frac{3}{2}\end{bmatrix}}{2(e^{\alpha})^{2\nu+2}}{}_2F_1\left(2\nu+2;\frac{1}{2};2\nu+\frac{5}{2};e^{-2\alpha}\right).
\end{aligned}\tag{3.4}$$

Proof: First, by the multiplying the relation (1.6) by itself and considering (3.1), we have

$$\begin{aligned}
&\Gamma^2(\nu-\mu+1)\,e^{-2\mu\pi i}[Q_\nu^{\mu}(\cosh(\alpha))]^2\\
&=\mathscr{L}\left\{\int_0^t K_\mu((t-\tau)\sinh(\alpha))K_\mu(\tau\sinh(\alpha))\,(t-\tau)^\nu\tau^\nu\,d\tau;\cosh(\alpha)\right\}.
\end{aligned}\tag{3.5}$$

Next, we change μ to $i\mu$ and integrate the both sides of the above relation with respect to μ. We employ the following addition formula for K_0 [7, p. 749, 6.791(4)]:

$$\frac{2}{\pi}\int_0^\infty K_{i\mu}(a)K_{i\mu}(b)\,d\mu=K_0(a+b),\tag{3.6}$$

and get

$$\begin{aligned}
&\int_0^\infty \Gamma^2(\nu-i\mu+1)\,e^{2\mu\pi i}[Q_\nu^{i\mu}(\cosh(\alpha))]^2\,d\mu\\
&=\frac{\pi}{2}\int_0^\infty e^{-t\cosh(\alpha)}K_0(t)\int_0^t(t-\tau)^{\nu-1}\tau^{\nu-1}\,d\tau\,dt\\
&=\frac{\pi}{2}B(\nu,\nu)\int_0^\infty t^{2\nu-1}\,e^{-t\cosh(\alpha)}K_0(t)\,dt.
\end{aligned}\tag{3.7}$$

Finally, by using the following Lipschitz–Hankel integral [12, p. 349, 3.16.1(3)]:

$$\begin{aligned}
\int_0^\infty e^{-ts}t^pK_\mu(at)\,dt&=\frac{(2a)^\mu\sqrt{\pi}}{(s+a)^{p+\mu+1}}\Gamma\begin{bmatrix}p-\mu+1,p+\mu+1\\ p+\frac{1}{2}\end{bmatrix}\\
&\times{}_2F_1\left(p+\mu+1,\mu+\frac{1}{2};p+\frac{3}{2};\frac{s-a}{s+a}\right),\\
&(\Re(p)>|\Re(\mu)|-1,\ \Re(s+a)>0),
\end{aligned}\tag{3.8}$$

the relation (3.4) is obtained. ■

Theorem 3.2: *For $\Re(\nu) > |\Re(\mu)|$, the following integral representation holds:*

$$\int_0^\infty \Gamma^2(\nu - i\mu)\Gamma^2(\nu + i\mu)\left[P_{i\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))\right]^2 \mathrm{d}\mu$$
$$= \frac{\sqrt{\pi}\,\Gamma\left[\begin{smallmatrix}\nu,\nu,2\nu\\ 2\nu-\frac{1}{2}\end{smallmatrix}\right]}{\sinh^{1-2\nu}(\alpha)(\cosh(\alpha)+1)^{2\nu}}\,{}_2F_1\left(2\nu, \frac{1}{2}; 2\nu + \frac{1}{2}; \frac{\cosh(\alpha)-1}{\cosh(\alpha)+1}\right). \tag{3.9}$$

Proof: To prove this theorem, we multiply the relation (1.8) by itself and consider (3.1) to get

$$\frac{\pi}{2\sinh^{2\nu-1}(\alpha)}\Gamma^2(\nu-\mu)\Gamma^2(\nu+\mu)\left[P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))\right]^2$$
$$= \mathscr{L}\left\{\int_0^t K_\mu(t-\tau)K_\mu(\tau)(t-\tau)^{\nu-1}\tau^{\nu-1}\,\mathrm{d}\tau; \cosh(\alpha)\right\}. \tag{3.10}$$

The remaining proof has a procedure similar to the proof of the previous theorem. ■

Theorem 3.3: *For $\Re(\nu+1) > |\Re(\mu)|$, the following integral representation holds for the products of associated Legendre functions:*

$$\frac{\mathrm{e}^{-\mu\pi\mathrm{i}}\Gamma(2\mu+1)\Gamma\left[\begin{smallmatrix}\nu-\mu+1\\ \nu+\mu+1\end{smallmatrix}\right]}{\sinh^{2\mu}(\alpha)}P_\nu^{-\mu}(\cosh(\alpha))Q_\nu^\mu(\cosh(\alpha))$$
$$= \int_0^\infty \frac{\sinh^{2\mu}(x)}{\left(\cosh^2(\alpha)-\sinh^2(\alpha)\cosh^2(x)\right)^{\nu+\mu+1}}$$
$$\times {}_2F_1\left(\nu+\mu+1, \nu+\mu+1; 2\mu+1; -\frac{\sinh^2(\alpha)\sinh^2(x)}{\cosh^2(\alpha)-\sinh^2(\alpha)\cosh^2(x)}\right)\mathrm{d}x. \tag{3.11}$$

Proof: We multiply the relations (1.5) and (1.6) and consider (3.2) to obtain

$$\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)\,\mathrm{e}^{-\mu\pi\mathrm{i}}P_\nu^{-\mu}(\cosh(\alpha))Q_\nu^\mu(\cosh(\alpha))$$
$$= \mathscr{M}\left\{\int_0^\infty \mathrm{e}^{-\frac{t}{\tau}\cosh(\alpha)-\tau\cosh(\alpha)}I_\mu\left(\frac{t}{\tau}\sinh(\alpha)\right)K_\mu(\tau\sinh(\alpha))\frac{\mathrm{d}\tau}{\tau}; \nu+1\right\}. \tag{3.12}$$

In the above relation, we insert the following representation for the product of the modified Bessel functions [7, p. 713, 6.656(1)]:

$$I_\mu(z)K_\mu(\zeta) = \int_0^\infty \mathrm{e}^{-(\zeta-z)\cosh(x)}J_{2\mu}(2\sqrt{z\zeta}\sinh(x))\,\mathrm{d}x, \quad \Re(\mu) > -\frac{1}{2},\ \Re(\zeta - z) > 0, \tag{3.13}$$

and take into account the integral (2.81) to get

$$\begin{aligned}&\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)\,\mathrm{e}^{-\mu\pi\mathrm{i}}P_\nu^{-\mu}(\cosh(\alpha))Q_\nu^{\mu}(\cosh(\alpha))\\&\quad=\Gamma\begin{bmatrix}\mu+\nu+1\\2\mu+1\end{bmatrix}\sinh^{2\mu}(\alpha)\int_0^\infty\frac{\sinh^{2\mu}(x)}{(\cosh(\alpha)-\sinh(\alpha)\cosh(x))^{\nu+\mu+1}}\\&\quad\times\int_0^\infty\frac{\tau^{\nu+\mu}{}_1F_1\left(\mu+\nu+1;2\mu+1;-\frac{\sinh^2(\alpha)\sinh^2(x)}{\cosh(\alpha)-\sinh(\alpha)\cosh(x)}\tau\right)}{\mathrm{e}^{\tau(\cosh(\alpha)+\sinh(\alpha)\cosh(x))}}\,\mathrm{d}\tau\,\mathrm{d}x.\end{aligned}$$

Finally, by using the Laplace transform of confluent hypergeometric function given in [12, p. 510,3.35.1(2)], we evaluate the inner integral of the above relation and get the result. ■

By the same procedure for the product of relations (1.7) and (1.8), we can write the following theorem for the product $Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh(\alpha))\,P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))$.

Theorem 3.4: *The following integral representation holds for* $\Re(\nu)>|\Re(\mu)|$:

$$\begin{aligned}&\frac{\mathrm{e}^{(\frac{1}{2}-\nu)\pi\mathrm{i}}\Gamma(2\mu+1)\Gamma\left[\begin{smallmatrix}\nu-\mu\\\nu+\mu\end{smallmatrix}\right]}{\sinh^{2\nu-1}(\alpha)}Q_{\mu-\frac{1}{2}}^{\nu-\frac{1}{2}}(\cosh(\alpha))P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))\\&\quad=\int_0^\infty\frac{\sinh^{2\mu}(x)}{\left(\cosh^2(\alpha)-\cosh^2(x)\right)^{\nu+\mu}}\\&\quad\times{}_2F_1\left(\nu+\mu,\nu+\mu;2\mu+1;-\frac{\sinh^2(x)}{\cosh^2(\alpha)-\cosh^2(x)}\right)\mathrm{d}x.\end{aligned}\tag{3.14}$$

Theorem 3.5: *The following integral representations hold for the products of associated Legendre functions:*

$$\begin{aligned}&\frac{\mathrm{e}^{-2\mu\pi\mathrm{i}}\Gamma(2\nu+2)\sinh^{2\nu+2}(\alpha)}{\cos(\mu\pi)\Gamma^2(\nu+\mu+1)}[Q_\nu^\mu(\cosh(\alpha))]^2\\&\quad=\int_0^\infty\frac{1}{\sinh^{2\nu+2}(x)}\\&\quad\times{}_2F_1\left(\nu+\mu+1,\nu-\mu+1;2\nu+2;\frac{\sinh^2(\alpha)\sinh^2(x)}{(\cosh(\alpha)+\sinh(\alpha)\cosh(x))^2}-1\right)\mathrm{d}x,\\&\quad\Re(\nu+1)>|\Re(\mu)|,\end{aligned}\tag{3.15}$$

$$\begin{aligned}&\frac{\pi\Gamma(2\nu)}{2\cos(\mu\pi)\sinh^{2\nu-1}(\alpha)}\left[P_{\mu-\frac{1}{2}}^{\frac{1}{2}-\nu}(\cosh(\alpha))\right]^2\\&\quad=\int_0^\infty\frac{1}{\sinh^{2\nu}(x)}\\&\quad\times{}_2F_1\left(\nu+\mu,\nu-\mu;2\nu;\frac{\sinh^2(x)}{(\cosh(\alpha)+\cosh(x))^2}-1\right)\mathrm{d}x,\quad\Re(\nu)>|\Re(\mu)|.\end{aligned}\tag{3.16}$$

Proof: We multiply the relation (1.6) by itself and consider (3.2) to write

$$\Gamma^2(\nu-\mu+1)\,\mathrm{e}^{-2\mu\pi\mathrm{i}}[Q_\nu^\mu(\cosh(\alpha))]^2 = \mathscr{M}\left\{\int_0^\infty \mathrm{e}^{-\frac{t}{\tau}\cosh(\alpha)-\tau\cosh(\alpha)} K_\mu\left(\frac{t}{\tau}\sinh(\alpha)\right)K_\mu(\tau\sinh(\alpha))\frac{\mathrm{d}\tau}{\tau};\nu+1\right\}. \quad (3.17)$$

We apply the following representation for the product of the modified Bessel functions [7, p. 713, 6.656(2)]:

$$K_\mu(z)K_\mu(\zeta) = 2\cos(\mu\pi)\int_0^\infty \mathrm{e}^{-(\zeta+z)\cosh(x)}K_{2\mu}(2\sqrt{z\zeta}\sinh(x))\mathrm{d}x,$$
$$|\Re(\mu)| < \tfrac{1}{2}, \quad \Re(\sqrt{z}+\sqrt{\zeta})^2 > 0, \quad (3.18)$$

and use the following Lipschitz–Hankel integral [12, p. 353,3.16.2(3)] to get the associated integral in terms of the Whittaker confluent hypergeometric function W

$$\int_0^\infty \mathrm{e}^{-ts}t^pK_\mu(a\sqrt{t})\,\mathrm{d}t = \frac{s^{-p-1/2}}{a}\Gamma\left(p+\frac{\mu}{2}+1\right)\Gamma\left(p-\frac{\mu}{2}+1\right) \times\exp\left(\frac{a^2}{8s}\right)W_{-p-1/2,\mu/2}\left(\frac{a^2}{4s}\right), \quad 2\Re(p) > |\Re(\mu)|-2, \quad \Re(s)>0. \quad (3.19)$$

Therefore, we rewrite (3.17) as

$$\Gamma^2(\nu-\mu+1)\,\mathrm{e}^{-2\mu\pi\mathrm{i}}[Q_\nu^\mu(\cosh(\alpha))]^2 = \frac{\cos(\mu\pi)\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)}{\sinh(\alpha)} \times\int_0^\infty \frac{\mathrm{e}^{-\tau\sinh(\alpha)\cosh(x)}}{\sinh(x)\,(\cosh(\alpha)+\sinh(\alpha)\cosh(x))^{\nu+\frac{1}{2}}} \times\int_0^\infty \frac{\tau^{\nu-\frac{1}{2}}}{\mathrm{e}^{\tau\cosh(\alpha)}}\,\mathrm{e}^{\frac{\tau\sinh^2(\alpha)\sinh^2(x)}{2(\cosh(\alpha)+\sinh(\alpha)\cosh(x))}} \times W_{-\nu-\frac{1}{2},\mu}\left(\frac{\sinh^2(\alpha)\sinh^2(x)}{\cosh(\alpha)+\sinh(\alpha)\cosh(x)}\tau\right)\mathrm{d}\tau\,\mathrm{d}x.$$

At this stage, by considering the following relation for the Tricomi confluent hypergeometric function [5, p. 274, 9.13(11)]:

$$W_{k,\mu}(z) = z^{\mu+1/2}\,\mathrm{e}^{-z/2}\Psi(\tfrac{1}{2}-k+\mu, 2\mu+1; z), \quad |\arg(z)| < \pi, \quad (3.20)$$

and evaluating the associated Laplace transform [12, p. 518,3.36.1(2)]

$$\int_0^\infty \mathrm{e}^{-ts} t^p \Psi(a,b;\omega t)\,\mathrm{d}t = \omega^{-p-1}\Gamma\begin{bmatrix} p+1, p-b+2 \\ p+a-b+2 \end{bmatrix} \times {}_2F_1\left(p+1, p-b+2; p+a-b+2; \frac{\omega-s}{s}\right), \quad \Re(p) > -1,\ \Re(p-b) > -2,\ \Re(s) > 0, \tag{3.21}$$

we get the result. Also, by multiplying the relation (1.8) by itself and applying the same procedure, the relation (3.16) is proved. ∎

Theorem 3.6: *The following identity holds for the quadratic product of toroidal functions:*

$$\sqrt{\frac{\pi}{2}}\frac{\pi^3}{\cosh^2(\pi\tau)}[P_{i\tau-\frac{1}{2}}(\cosh(\alpha))]^2 = \mathscr{K}\mathscr{L}\left\{\frac{\mathrm{e}^{t\cosh^2(\alpha)}}{\sqrt{t}}\int_0^{\frac{\pi}{2}} K_0\left(2t\left(2\csc^2 2\theta\cosh^2(\alpha) - \frac{\sinh^2(\alpha)}{2}\right)\right)\mathrm{d}\theta;\tau\right\}. \tag{3.22}$$

Proof: We begin with the relation

$$P_{\mu-\frac{1}{2}}(\cosh(\alpha)) = \frac{\cos(\mu\pi)\sqrt{\frac{2}{\pi}}}{\pi}\int_0^\infty \mathrm{e}^{-t\cosh(\alpha)}K_\mu(t)\frac{\mathrm{d}t}{\sqrt{t}}, \tag{3.23}$$

and change μ to $i\tau$. Then, we use (3.3) as the convolution product of the Kontorovich-Lebedev transform to get

$$\frac{\pi^3}{2\cosh^2(\pi\tau)}[P_{i\tau-\frac{1}{2}}(\cosh(\alpha))]^2 = \mathscr{K}\mathscr{L}\left\{\frac{1}{2t}\int_0^\infty\int_0^\infty \mathrm{e}^{-\frac{1}{2}(\frac{v}{u}+\frac{u}{v})t-\frac{uv}{2t}}\,\mathrm{e}^{-(u+v)\cosh(\alpha)}\frac{\mathrm{d}u\mathrm{d}v}{\sqrt{uv}};\tau\right\} = \int_0^\infty K_{i\tau}(t)\frac{\mathrm{d}t}{2t}\int_0^\infty\int_0^\infty \mathrm{e}^{-\frac{1}{2}(\frac{v}{u}+\frac{u}{v})t-\frac{uv}{2t}}\,\mathrm{e}^{-(u+v)\cosh(\alpha)}\frac{\mathrm{d}u\,\mathrm{d}v}{\sqrt{uv}}. \tag{3.24}$$

We change of variables $uv = a$ and $\frac{u}{v} = b$, and compute the Jacobian $|\frac{\partial(a,b)}{\partial(u,v)}| = \frac{1}{2b}$ to rewrite (3.24) as

$$\frac{\pi^3}{2\cosh^2(\pi\tau)}[P_{i\tau-\frac{1}{2}}(\cosh(\alpha))]^2 = \mathscr{K}\mathscr{L}\left\{\frac{1}{4t}\int_0^\infty\int_0^\infty \mathrm{e}^{-\frac{1}{2}(b+\frac{1}{b})t-\frac{a}{2t}}\,\mathrm{e}^{-(\sqrt{ab}+\sqrt{\frac{a}{b}})\cosh(\alpha)}\frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{ab}};\tau\right\}. \tag{3.25}$$

We here use (2.23) and (2.26) to evaluate the inner integral in terms of the complementary error function

$$\int_0^\infty e^{-\frac{a}{2t}}\, e^{-(\sqrt{b}+\sqrt{\frac{1}{b}})\sqrt{a}\cosh(\alpha)}\,\frac{da}{\sqrt{a}}$$
$$= 2\sqrt{t}\, e^{\frac{1}{4}(\sqrt{b}+\frac{1}{\sqrt{b}})^2 t\cosh^2(\alpha)} D_{-1}\left(\left(\sqrt{b}+\frac{1}{\sqrt{b}}\right)\cosh(\alpha)\sqrt{t}\right)$$
$$= 2\sqrt{\frac{\pi}{2}}\sqrt{t}\, e^{\frac{1}{2}(\sqrt{b}+\frac{1}{\sqrt{b}})^2 t\cosh^2(\alpha)}\mathrm{Erfc}\left(\left(\sqrt{b}+\frac{1}{\sqrt{b}}\right)\cosh(\alpha)\sqrt{\frac{t}{2}}\right). \tag{3.26}$$

At this point, using the addition formula for the complementary error function [15]

$$\mathrm{Erfc}(z+\zeta) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} e^{-4\csc^2(2\theta)(z^2+\zeta^2)}\, d\theta, \quad |\arg(z)| < \frac{\pi}{2}, \quad |\arg(\zeta)| < \frac{\pi}{2}, \tag{3.27}$$

we can get (3.25) as

$$\frac{\pi^3}{2\cosh^2(\pi\tau)}[P_{i\tau-\frac{1}{2}}(\cosh(\alpha))]^2$$
$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2}}\mathscr{K}\mathscr{L}\left\{\frac{e^{t\cosh^2(\alpha)}}{\sqrt{t}}\right.$$
$$\left.\times \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-bt(\frac{1}{2}-\frac{\cosh^2(\alpha)}{2}+2\csc^2 2\theta\cosh^2(\alpha))-\frac{t}{b}(\frac{1}{2}-\frac{\cosh^2(\alpha)}{2}+2\csc^2 2\theta\cosh^2(\alpha))}\frac{db}{b}\, d\theta; \tau\right\}, \tag{3.28}$$

or equivalently

$$\frac{\pi^3}{2\cosh^2(\pi\tau)}[P_{i\tau-\frac{1}{2}}(\cosh(\alpha))]^2$$
$$= \sqrt{\frac{1}{2\pi}}\mathscr{K}\mathscr{L}\left\{\frac{e^{t\cosh^2(\alpha)}}{\sqrt{t}}\int_0^{\frac{\pi}{2}} K_0\left(2t\left(2\csc^2 2\theta\cosh^2(\alpha) - \frac{\sinh^2(\alpha)}{2}\right)\right) d\theta; \tau\right\}, \tag{3.29}$$

where we used the identity

$$2K_0(2\sqrt{pq}) = \int_0^\infty e^{-pb-\frac{q}{b}}\frac{db}{b}, \quad p, q > 0. \tag{3.30}$$

■

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